Lecture 17: Synchronization and the Kuramoto Model

From Vicsek alignment to phase coherence

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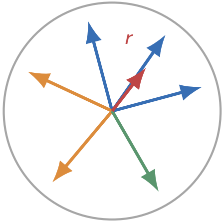
Prelecture question

In the Vicsek model each particle repeatedly averages the directions of nearby particles.

Now remove the positions. Suppose each agent only has a phase

$$\theta_i(t) \in [0, 2\pi),$$

and suppose the agents try to adjust their phases to nearby phases.



average phase vector

Question

Can a population with different intrinsic frequencies spontaneously synchronize?

Today we study the simplest mathematical model where this transition can be seen clearly.

Where we are in the course

We have already seen several mechanisms for collective behavior.

Model family	Local rule	Global question
Networks	edges constrain interaction	who influences whom?
Markov chains	probability is redistributed	where is mass in the long run?
Vicsek model	align with nearby headings	does the swarm polarize?
Persistent homology	compare structure across scales	which patterns persist?

Today we isolate one ingredient of Vicsek: alignment of angles.

The resulting model is not about moving particles. It is about synchronization of phases.

Lecture goals

In this lecture we will:

- formulate the Kuramoto model for coupled phase oscillators;
- interpret phase, natural frequency, coupling strength, and synchronization;
- define the complex order parameter;
- connect the Kuramoto order parameter with the Vicsek polarization;
- analyze the two-oscillator case exactly;
- explain the mean-field synchronization transition;
- simulate the model and measure the transition numerically;
- extend the model from all-to-all coupling to network coupling.

From Flocking to Synchronization

Alignment of moving directions becomes alignment of phases

Vicsek recap

In the Vicsek model each particle has a position and a heading angle:

$$x_i(t) \in [0,1)^2, \qquad \theta_i(t) \in [0, 2\pi).$$

The update rule is

$$\theta_i(t+1) = \arg\left(\sum_{j:\, d_{\mathrm{per}}(x_i(t), x_j(t)) < R} \mathrm{e}^{\mathrm{i}\theta_j(t)} \right) + \eta \xi_i(t),$$

$$x_i(t+1) = x_i(t) + v_0(\cos\theta_i(t+1), \sin\theta_i(t+1)) \mod 1.$$

The order parameter is

$$\Phi(t) = \left| \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} \right|.$$

What part do we keep?

Vicsek contains several modeling ingredients.

Ingredient	In the Vicsek model
State	position x_i and heading θ_i
Interaction	nearby particles in physical space
Alignment	average nearby directions on the unit circle
Motion	self-propulsion with speed v_0
Noise	angular perturbation of size η

In the Kuramoto model we keep the circular variable and the alignment mechanism, but remove the positions.

This makes the model simpler and more analytically transparent.

Examples of synchronization

Synchronization means that many units coordinate their phases or rhythms.

Examples include:

- fireflies flashing in synchrony;
- metronomes coupled through a moving support;
- pacemaker cells in cardiac tissue;
- circadian rhythms entrained by light-dark cycles;
- power-grid generators maintaining a common frequency;
- neurons whose firing phases become partially locked.

The mathematical question is not whether all these systems are identical.

It is whether a common reduced mechanism explains the transition from incoherence to coherence.

Independent phase oscillators

The simplest oscillator is a point moving around the unit circle.

$$\dot{\theta}_i = \omega_i, \qquad \theta_i(t) = \theta_i(0) + \omega_i t.$$

Here ω_i is the natural frequency of oscillator i.

If $\omega_i \neq \omega_j$, then the phase difference

$$\theta_j(t) - \theta_i(t)$$

usually drifts over time.

Without coupling, heterogeneous oscillators do not synchronize.

How should phases interact?

Suppose oscillator j pulls oscillator i toward its own phase.
The correction should satisfy three basic requirements:

- it depends only on the phase difference $\theta_j - \theta_i$;
- it is periodic with period 2π;
- it changes sign when the phase difference changes sign.

The simplest smooth choice is

$$\sin(\theta_j - \theta_i).$$

For small phase differences,

$$\sin(\theta_j - \theta_i) \approx \theta_j - \theta_i,$$

so the interaction resembles linear attraction.

The Kuramoto Model

A minimal model for spontaneous synchronization

All-to-all Kuramoto model

The classical Kuramoto model is

$$\boxed{\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad i = 1, \ldots, N.}$$

The variables and parameters are:

- $\theta_i \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$: phase of oscillator i;
- $\omega_i \in \mathbb{R}$: natural frequency;
- $K \geq 0$: coupling strength;
- $1/N$: normalization that keeps the total input comparable when N changes.

This is an ordinary differential equation on an N-dimensional torus.

Reading the equation

For oscillator i,

$$\dot{\theta}_i = \underbrace{\omega_i}_{\text{intrinsic drift}} + \underbrace{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}_{\text{social or physical coupling}} .$$

If θ_j is slightly ahead of θ_i, then

$$\sin(\theta_j - \theta_i) > 0,$$

so oscillator i speeds up.

If θ_j is slightly behind, oscillator i slows down.

Coupling therefore tries to reduce phase differences, while the natural frequencies try to pull oscillators apart.

Rotating frame

The absolute value of phase is often less important than phase differences.

Let

$$\bar{\omega} = \frac{1}{N} \sum_{i=1}^{N} \omega_i.$$

If we define new phases

$$\varphi_i(t) = \theta_i(t) - \bar{\omega} t,$$

then the new natural frequencies are

$$\omega_i - \bar{\omega}.$$

Thus we can often assume, after changing reference frame, that the average frequency is zero:

$$\frac{1}{N} \sum_i \omega_i = 0.$$

Identical oscillators

First suppose all natural frequencies are equal.

After moving to a rotating frame, this means

$$\omega_i = 0 \qquad \text{for all } i.$$

Then the model becomes

$$\dot{\theta}_i = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i).$$

The fully synchronized states

$$\theta_1 = \theta_2 = \cdots = \theta_N$$

are equilibria.

For attractive coupling, nearby phases tend to collapse toward a common phase.

Two oscillators: exact phase locking

For two oscillators,

$$\dot{\theta}_1 = \omega_1 + \frac{K}{2}\sin(\theta_2 - \theta_1),$$
$$\dot{\theta}_2 = \omega_2 + \frac{K}{2}\sin(\theta_1 - \theta_2).$$

Let $\delta = \theta_2 - \theta_1$ and $\Delta\omega = \omega_2 - \omega_1$. Then

$$\dot{\delta} = \Delta\omega - K\sin\delta.$$

A phase-locked solution has $\dot{\delta} = 0$, so

$$\sin\delta^* = \frac{\Delta\omega}{K}.$$

Therefore two oscillators can lock only if

$$|\Delta\omega| \leq K.$$

What phase locking means

Phase locking does not require equal phases.

It means the phase difference converges to a constant:

$$\theta_2(t) - \theta_1(t) \to \delta^*.$$

Then both oscillators rotate with the same observed frequency:

$$\dot{\theta}_1(t) \approx \dot{\theta}_2(t).$$

In heterogeneous populations, synchronization usually means frequency synchronization or partial phase coherence, not necessarily

$$\theta_1 = \cdots = \theta_N.$$

This distinction is important when interpreting simulations.

Order Parameter and Transition

Measuring coherence with one complex number

The complex order parameter

Define

$$\boxed{z(t) = r(t)\mathrm{e}^{\mathrm{i}\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)}.}$$

Here:

- $r(t) = |z(t)| \in [0, 1]$ measures phase coherence;
- $\psi(t) = \arg z(t)$ is the average phase direction.

Interpretation:

$$r \approx 0 \quad \Longleftrightarrow \quad \text{phases spread around the circle,}$$
$$r \approx 1 \quad \Longleftrightarrow \quad \text{phases concentrated near one direction.}$$

Same order parameter as Vicsek

The Vicsek polarization was

$$\Phi(t) = \left| \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} \right|.$$

The Kuramoto coherence is

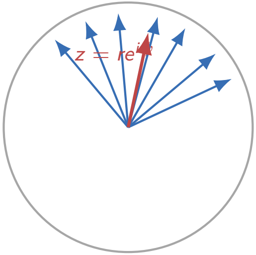
$$r(t) = \left| \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} \right|.$$

The same mathematical object measures order in two different models.

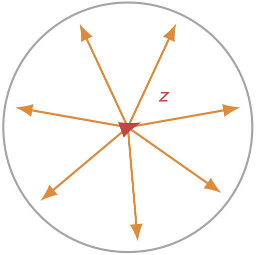
Modeling lesson

An order parameter should be chosen to measure the macroscopic pattern that the model is supposed to produce.

Geometric interpretation



coherent phases: large r



spread phases: small r

Mean-field rewriting

The order parameter lets us rewrite the coupling term.

Since

$$r\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j},$$

we have

$$r\sin(\psi - \theta_i) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i).$$

Therefore

$$\boxed{\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i).}$$

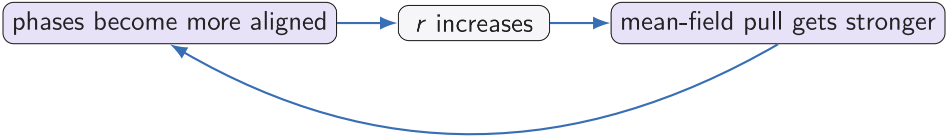
Each oscillator feels attraction toward the current mean phase, with effective strength Kr.

Positive feedback

The mean-field form

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i)$$

shows a feedback loop.



If K is too small, frequency heterogeneity dominates.

If K is large enough, some oscillators lock to the mean field and increase r.

The synchronization transition

Assume the natural frequencies are sampled from a symmetric density $g(\omega)$ centered at 0.

In the infinite-population all-to-all model, the incoherent state loses stability at

$$\boxed{K_c = \frac{2}{\pi g(0)}}.$$

For $K < K_c$, the population is mostly incoherent:

$$r \approx 0.$$

For $K > K_c$, a synchronized cluster appears:

$$r > 0.$$

This is a phase transition in a deterministic dynamical system with many interacting components.

Interpret the formula carefully

The critical coupling

$$K_c = \frac{2}{\pi g(0)}$$

is a theorem for the classical mean-field limit under specific assumptions.

It is not a universal formula for every synchronization problem.

Things that change the threshold include:

- finite population size;
- non-symmetric or multimodal frequency distributions;
- network topology instead of all-to-all coupling;
- time delays;
- noise;
- higher-order interactions.

The formula is still useful because it shows how heterogeneity and coupling compete.

Simulation and Networks

Numerically measuring synchronization and topology effects

Euler simulation

For a time step Δt, the explicit Euler method gives

$$\theta_i^{n+1} = \theta_i^n + \Delta t \left(\omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j^n - \theta_i^n) \right).$$

Then wrap phases modulo 2π:

$$\theta_i^{n+1} \leftarrow \theta_i^{n+1} \mod 2\pi.$$

At every time step compute

$$r^n = \left| \frac{1}{N} \sum_j \mathrm{e}^{\mathrm{i}\theta_j^n} \right|.$$

The most important numerical output is not a single trajectory, but $r(t)$ and its long-time average.

Numerical experiment

A simple experiment is:

1. choose N, for example $N = 200$;
2. sample ω_i from a normal distribution and subtract the mean;
3. sample initial phases uniformly in $[0, 2\pi)$;
4. choose a list of coupling strengths K;
5. simulate long enough to remove transients;
6. plot the time-averaged order parameter $\langle r \rangle$ versus K.

Expected behavior:

- small K: phases drift almost independently;
- intermediate K: partial synchronization;
- large K: most phases rotate coherently.

What can go wrong numerically?

The Kuramoto model is simple, but simulations still require care.

Common issues:

- too large a time step creates numerical artifacts;
- too short a simulation confuses transients with asymptotic behavior;
- one random initial condition may not represent typical behavior;
- finite-size fluctuations make r nonzero even below threshold;
- plotting phases without wrapping can obscure the circular geometry.

Validation checks:

- verify $0 \leq r(t) \leq 1$;
- repeat with different seeds;
- decrease Δt and compare;
- test the identical-frequency case, where synchronization should be easy.

Kuramoto on a network

In many systems, each oscillator does not interact with all others.

Let A be the adjacency matrix of a graph. A network Kuramoto model is

$$\boxed{\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i).}$$

Sometimes one uses a normalized form, for example

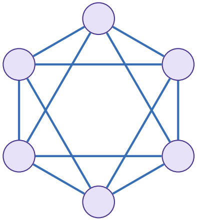
$$\dot{\theta}_i = \omega_i + \frac{K}{d_i} \sum_{j} A_{ij} \sin(\theta_j - \theta_i),$$

where $d_i = \sum_j A_{ij}$.

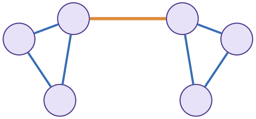
The choice depends on whether high-degree nodes should receive more total input or comparable total input.

Topology matters

The network affects synchronization through paths, bottlenecks, communities, and degree heterogeneity.



dense graph: many paths



two communities: one bottleneck

Strong internal coupling can synchronize communities before the whole graph synchronizes.

Linearization near synchrony

Suppose phases are close, so

$$\sin(\theta_j - \theta_i) \approx \theta_j - \theta_i.$$

For identical frequencies on a graph,

$$\dot{\theta}_i \approx K \sum_j A_{ij}(\theta_j - \theta_i).$$

Using the graph Laplacian $L = D - A$, this becomes

$$\dot{\theta} \approx -KL\theta.$$

This is the same mathematical structure as diffusion and consensus on networks.

Lecture 18 will take this linear averaging mechanism as the starting point.

Power-grid interpretation

The Kuramoto model is often used as a reduced model for coupled AC generators.

In a power grid:

- phase differences determine power flows;
- generators must maintain a common frequency;
- network topology and line capacities constrain synchronization;
- loss of synchrony can produce cascading failures.

Real grid models include inertia, voltage, control, and constraints.

The Kuramoto model is not the full engineering model, but it captures the key competition between natural frequencies, coupling, and topology.

In-class exercise

Consider the two-oscillator phase-difference equation

$$\dot{\delta} = \Delta\omega - K \sin \delta.$$

Answer the following:

1. For which values of K does a phase-locked equilibrium exist?
2. If it exists, which equilibrium is stable?
3. What happens when K is below the locking threshold?
4. How would you detect locking numerically from $\theta_1(t)$ and $\theta_2(t)$?

Hint: examine

$$f(\delta) = \Delta\omega - K \sin \delta$$

and the sign of $f'(\delta)$ at an equilibrium.

Summary

Today we introduced the Kuramoto model

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_j \sin(\theta_j - \theta_i).$$

Main points:

- phase synchronization is a collective effect produced by local or global coupling;
- the order parameter $r = \left| N^{-1} \sum_j \mathrm{e}^{\mathrm{i}\theta_j} \right|$ measures coherence;
- the same complex average appears in the Vicsek polarization;
- two oscillators lock when coupling overcomes frequency mismatch;
- in large mean-field populations, synchronization appears through a transition in K;
- on networks, topology affects how synchronization spreads.

Next time: consensus and flocking as graph-based averaging dynamics.