Lecture 14: From Networks to Holes

Vietoris–Rips Complexes and Homology over $\mathbb{Z}_2$

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Prelecture question

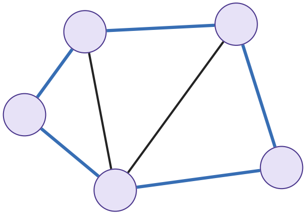
We place sensors in a field and want to know whether their sensing regions cover the whole field.

We do not observe the field directly. Instead, we are given a detection graph.

For a fixed tunable sensing radius R, connect sensors A and B when

$$d(A, B) \leq R.$$

Suppose the resulting graph contains a cycle.



cycle in the detection graph

Question

If we see a cycle in the graph, can we conclude that the sensors do not cover the entire field?

Today we will see why graph cycles alone are not enough, and how homology distinguishes filled cycles from genuine coverage holes.

Lecture goals

In this lecture we will:

- distinguish the embedded sensor picture from the abstract detection graph;
- turn pairwise network data into a simplicial complex;
- define the Vietoris–Rips complex as a filled clique complex;
- represent simplices as vector spaces over $\mathbb{Z}_2$;
- define boundary maps ∂_1 and ∂_2;
- explain why $\partial_1 \partial_2 = 0$;
- define cycles, boundaries, and homology;
- compute β_1 using ranks of boundary matrices;
- distinguish graph cycles from topological holes.

What information do we actually have?

We imagine sensors located in a region of the plane.

But in the data problem, their coordinates are not observed.

For a chosen detection radius R, the available information is only pairwise detection:

$$\{i, j\} \in E_R \quad \Longleftrightarrow \quad \text{sensor } i \text{ and sensor } j \text{ detect each other at radius } R.$$

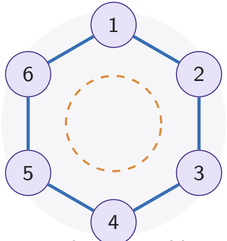
Thus the object we can construct is the Vietoris–Rips graph

$$G_R = (V, E_R).$$

Changing R changes the graph: edges, cycles, and cliques may appear or disappear.

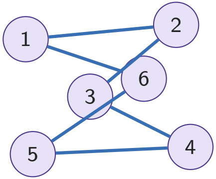
The same graph, two very different views

Embedded picture



coordinates visible;
central empty region visible

Abstract graph



same vertices and edges;
no coordinates or distances

Important distinction

The two drawings represent the same graph combinatorially. Only the embedded picture contains geometric information.

Why the hole is not obvious

In the embedded picture, a hole is a visual statement about empty space in the plane.

In the abstract Vietoris–Rips graph, there is no notion of inside, outside, area, or planar position. There are only vertices and pairwise detections.

Therefore the question cannot be answered by looking for graph cycles alone:

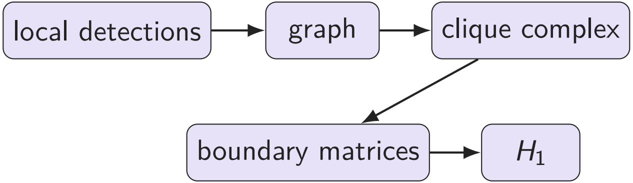
- some cycles surround genuine uncovered regions;
- some cycles are filled by triangles coming from mutually detecting triples;
- the drawing of the abstract graph may hide the geometric intuition completely.

The next step is to fill all cliques:

$$G_R \quad \longmapsto \quad \mathrm{Cl}(G_R) = \mathrm{VR}_R \,.$$

Homology will decide which cycles remain after those fillings.

The central message



Complex-systems theme

From local pairwise information, we try to infer global structure.

The mathematical tool will be homology, presented as a rank computation.

From Sensors to Complexes

Fill mutually connected groups

The sensor-network problem

We deploy sensors in a region.

Each sensor detects nearby sensors, but we do not necessarily know exact coordinates.

The available data is a graph

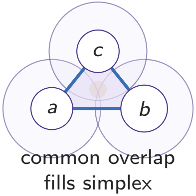
$$G = (V, E),$$

where

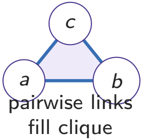
$$(i, j) \in E \iff \text{sensor } i \text{ detects sensor } j.$$

Question: can this local graph reveal global loops or holes?

Čech complex

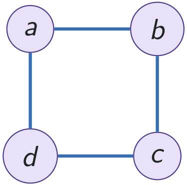


Vietoris–Rips complex



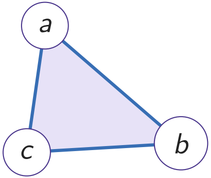
Graphs alone are not enough

Square cycle



looks like a hole

Triangle cycle



cycle, but filled

First slogan

A graph cycle is not necessarily a topological hole.

Clique complex of a graph

Given a graph G, its clique complex is denoted

$$\mathrm{Cl}(G).$$

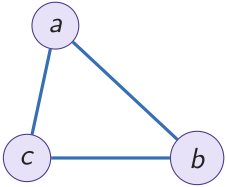
A set of vertices spans a simplex if every pair of vertices in that set is connected by an edge.

Graph structure	Simplex in $\mathrm{Cl}(G)$
vertex	0-simplex
edge	1-simplex
triangle clique	2-simplex
4-clique	3-simplex
$(k+1)$-clique	k-simplex

The clique complex fills every mutually connected group.

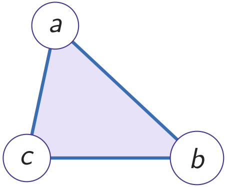
Triangle graph versus filled triangle

As a graph



$ab + bc + ca$

As a clique complex



$[a, b, c]$ is filled

The cycle becomes the boundary of a filled simplex, so it should not count as a hole.

Vietoris–Rips complex

Let

$$X = \{x_1, \ldots, x_n\}$$

be a finite metric space or point cloud, and let $r > 0$.

The Vietoris–Rips complex $\mathrm{VR}(X, r)$ contains a simplex

$$[x_{i_0}, \ldots, x_{i_k}]$$

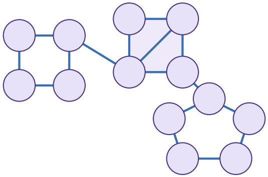
whenever all pairwise distances satisfy

$$d(x_{i_a}, x_{i_b}) \le r.$$

Equivalently:

1. build the graph with edges between points at distance at most r;
2. fill all cliques in that graph.

One scale, one complex



one threshold graph plus filled cliques

At a fixed scale r we get one simplicial complex.

The complex contains:

- vertices: sensors or data points;
- edges: pairwise detections or close pairs;
- triangles: mutually connected triples;
- higher simplices: larger mutually connected groups.

Today we study homology of one fixed complex.

Chains and Boundaries

Turn pictures into vector spaces

Chain groups over $\mathbb{Z}_2$

For each dimension k, define

$$C_k = \text{the vector space over } \mathbb{Z}_2 \text{ generated by the } k\text{-simplices.}$$

The field is

$$\mathbb{Z}_2 = \{0, 1\}, \qquad 1 + 1 = 0.$$

Over $\mathbb{Z}_2$, a chain is simply a subset of simplices:

- a 0-chain is a subset of vertices;
- a 1-chain is a subset of edges;
- a 2-chain is a subset of filled triangles.

Addition is symmetric difference: objects appearing twice cancel.

Example: a chain as a vector

Suppose the edges are ordered as

$$e_1, e_2, e_3, e_4, \qquad C_1 \cong \mathbb{Z}_2^4.$$

The chain

$$e_1 + e_3 + e_4$$

is represented by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Visually it means: select exactly the edges e_1, e_3, e_4.

Addition of chains is like listing selected simplices, with cancellation modulo 2:

$$e_1 + e_1 = 0, \qquad e_1 + e_3 + e_1 = e_3.$$

Boundary maps

We define linear maps

$$\partial_k : C_k \to C_{k-1}.$$

The two maps we need today are

$$\partial_1 : C_1 \to C_0, \qquad \partial_2 : C_2 \to C_1.$$

Over $\mathbb{Z}_2$:

$$\partial_1([a, b]) = [a] + [b],$$

and

$$\partial_2([a, b, c]) = [a, b] + [a, c] + [b, c].$$

No orientations and no signs are needed in this first treatment.

Boundary of an edge

The edge is a 1-simplex:

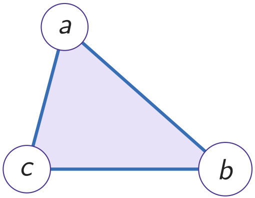
$$[a, b] \in C_1.$$

Its boundary is the set of endpoints:

$$\partial_1([a, b]) = [a] + [b].$$

In matrix language, ∂_1 maps selected edges to their incident vertices modulo 2.

Boundary of a triangle



The filled triangle is a 2-simplex:

$$[a, b, c] \in C_2.$$

Its boundary is the surrounding edge cycle:

$$\partial_2([a, b, c]) = [a, b] + [a, c] + [b, c].$$

This is the algebraic reason that a filled triangle does not contribute an H_1 hole.

Boundary of boundary equals zero

Compute the boundary of the boundary of a triangle:

$$\begin{aligned}\partial_1\partial_2([a,b,c]) &= \partial_1([a,b]+[a,c]+[b,c]) \\ &= ([a]+[b])+([a]+[c])+([b]+[c]) \\ &= 0.\end{aligned}$$

Each vertex appears twice and cancels modulo 2.

Thus

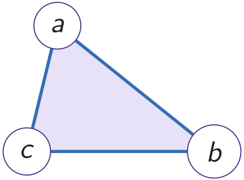
$$\partial_1\partial_2 = 0.$$

Geometric slogan

The boundary of a boundary is zero.

Boundaries of boundaries are empty

Filled triangle



edges form the boundary

The edge boundary has no remaining boundary: each endpoint appears twice.

Segment

endpoints form the boundary

The endpoints themselves have no further boundary.

Geometric principle

Taking the boundary twice gives the empty chain.

Homology

Cycles modulo boundaries

Cycles

The k-cycles are

$$Z_k = \ker \partial_k.$$

They are chains with zero boundary.

For $k = 1$, a 1-cycle is an edge chain in which every vertex is incident to an even number of selected edges.

In pictures, these are closed edge loops, or sums of closed loops.

Linear algebra translation

Finding cycles means solving $\partial_k z = 0$.

Boundaries

The k-boundaries are

$$B_k = \operatorname{im} \partial_{k+1}.$$

They are cycles that arise as boundaries of higher-dimensional chains.

For $k = 1$, these are boundaries of filled triangles or sums of filled triangles.

Because

$$\partial_k \partial_{k+1} = 0,$$

we have

$$B_k \subseteq Z_k.$$

So every boundary is automatically a cycle.

Fundamental question

From

$$\partial_k \partial_{k+1} = 0$$

we know that

$$B_k \subseteq Z_k.$$

Every boundary is a cycle.

Fundamental question

Are there cycles that are not boundaries?

This is where holes enter. If a filled face is present, its edge loop is a boundary.

If the face is absent and only its edge loop is present, that loop is not the boundary of a 2-simplex in the complex; it is just a cycle.

Homology groups

Homology is defined by

$$H_k = Z_k / B_k.$$

This does not mean removing the boundaries from the cycles.

It means:

Quotient interpretation

Declare boundaries to be zero and identify cycles that differ by a boundary.

For H_1:

$$H_1 = \frac{\text{closed edge cycles}}{\text{boundaries of filled 2-chains}}.$$

H_1 detects cycles that are not explained as boundaries of filled triangles.

Quotient intuition over $\mathbb{Z}_2$

Two cycles $z, z' \in Z_k$ represent the same homology class if

$$z + z' \in B_k.$$

This is the same as saying that z and z' differ by a boundary, because subtraction and addition are identical over $\mathbb{Z}_2$.

Statement	Meaning
$z \in Z_k$	z is closed
$z \in B_k$	z bounds something filled
$[z] = [z']$	z and z' differ by a boundary

Homology classes are equivalence classes of cycles, not just individual cycles.

Betti numbers

The k-th Betti number is the dimension of homology:

$$\beta_k = \dim H_k.$$

Since $H_k = Z_k / B_k$,

$$\beta_k = \dim Z_k - \dim B_k.$$

Using rank-nullity:

$$\dim Z_k = \dim C_k - \operatorname{rank}(\partial_k), \qquad \dim B_k = \operatorname{rank}(\partial_{k+1}).$$

Therefore

$$\boxed{\beta_k = \dim C_k - \operatorname{rank}(\partial_k) - \operatorname{rank}(\partial_{k+1}).}$$

The formula for β_1

For one-dimensional holes:

$$\boxed{\beta_1 = \dim C_1 - \operatorname{rank}(\partial_1) - \operatorname{rank}(\partial_2).}$$

Interpretation:

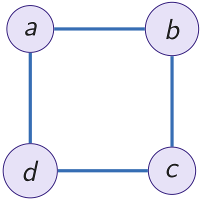
- $\dim C_1$ counts available edges;
- $\operatorname{rank}(\partial_1)$ removes edge chains that are not cycles;
- $\operatorname{rank}(\partial_2)$ removes cycles that are already filled by triangles.

All ranks are computed by Gaussian elimination over $\mathbb{Z}_2$.

Worked Examples

Square, filled triangle, annulus

Example 1: square cycle



no filled triangles

There are four vertices and four edges:

$$\dim C_0 = 4, \qquad \dim C_1 = 4, \qquad \dim C_2 = 0.$$

The graph is connected, so

$$\operatorname{rank}(\partial_1) = 3.$$

Since $C_2 = 0$,

$$\operatorname{rank}(\partial_2) = 0.$$

Therefore

$$\beta_1 = 4 - 3 - 0 = 1.$$

Example 1: boundary matrix

Order the vertices and edges as

$$a, b, c, d, \qquad ab, bc, cd, da.$$

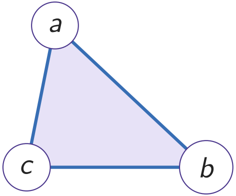
Then

$$\partial_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Gaussian elimination over $\mathbb{Z}_2$ gives $\text{rank}(\partial_1) = 3$.

Thus the nullity of ∂_1 is $4 - 3 = 1$: there is one independent edge cycle.

Example 2: filled triangle



There are three vertices, three edges, and one filled triangle:

$$\dim C_0 = 3, \quad \dim C_1 = 3, \quad \dim C_2 = 1.$$

The graph is connected, so

$$\mathrm{rank}(\partial_1) = 2.$$

The triangle has nonzero boundary, so

$$\mathrm{rank}(\partial_2) = 1.$$

Therefore

$$\beta_1 = 3 - 2 - 1 = 0.$$

The lesson from square and triangle

Complex	dim C_1	rank ∂_1	rank ∂_2
Square cycle	4	3	0
Filled triangle	3	2	1

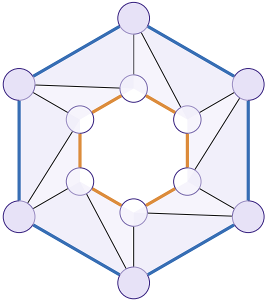
Therefore

$$\beta_1(\text{square}) = 1, \qquad \beta_1(\text{filled triangle}) = 0.$$

Second slogan

A hole is a cycle that is not a boundary.

Example 3: annulus and quotienting



two cycles around the same hole

Let

$$z_1 = \text{outer boundary cycle},$$
$$z_2 = \text{inner boundary cycle}.$$

Both are cycles:

$$\partial z_1 = 0, \qquad \partial z_2 = 0.$$

If A is the sum of the filled triangles in the annular strip, then

$$\partial A = z_1 + z_2.$$

Homologous cycles in the annulus

From the annulus computation,

$$z_1 + z_2 = \partial A.$$

Thus

$$z_1 + z_2 \in B_1.$$

Therefore

$$[z_1] = [z_2] \in H_1.$$

The two cycles are different edge chains, but they represent the same hole.

This is why homology is a quotient: it groups cycles that enclose the same hole.

Exercise

Consider four vertices $0, 1, 2, 3$ with:

- the four square edges $01, 12, 23, 30$;
- one diagonal 02;
- two filled triangles 012 and 023.

Questions:

1. What are $\dim C_0$, $\dim C_1$, and $\dim C_2$?
2. What is $\operatorname{rank}(\partial_1)$?
3. What is $\operatorname{rank}(\partial_2)$?
4. What is β_1?

Geometric check: should the filled square have a one-dimensional hole?

What to remember

- A Vietoris–Rips complex is a threshold graph with all cliques filled.
- Over $\mathbb{Z}_2$, chains are subsets of simplices.
- Boundary maps turn topology into linear algebra.
- Cycles are kernels: $Z_k = \ker \partial_k$.
- Boundaries are images: $B_k = \operatorname{im} \partial_{k+1}$.
- Homology is cycles modulo boundaries: $H_k = Z_k / B_k$.
- $\beta_1 = \dim C_1 - \operatorname{rank}(\partial_1) - \operatorname{rank}(\partial_2)$.
- A graph cycle is not necessarily a topological hole.

Bridge to Lecture 15

Today we studied homology at one fixed scale r.

But in data analysis, the correct scale is often unknown.

For a point cloud or weighted network, we can increase the threshold:

$$r_1 \leq r_2 \quad \Longrightarrow \quad \mathrm{VR}(X, r_1) \subseteq \mathrm{VR}(X, r_2).$$

Lecture 15 will study which holes appear and which holes survive as r varies.

Next slogan

Persistent homology tracks which holes survive across scales.

Suggested bibliography

For students who want to read more:

- H. Edelsbrunner and J. Harer, *Computational Topology: An Introduction*, American Mathematical Society, 2010.
- G. Carlsson, "Topology and data," *Bulletin of the AMS*, 46(2), 255–308, 2009.
- R. Ghrist, "Barcodes: The persistent topology of data," *Bulletin of the AMS*, 45, 61–75, 2008.
- A. Hatcher, *Algebraic Topology*, Chapter 2, for a standard reference on homology.
- GUDHI documentation: `https://gudhi.inria.fr/`.

Acknowledgment

These slides were prepared with the assistance of OpenAI Codex / ChatGPT 5.5 as a drafting and editing tool.
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