# Lecture 6: Chaos, Sensitivity, and Predictability

## Modelling Complex Systems (Spring 2026)

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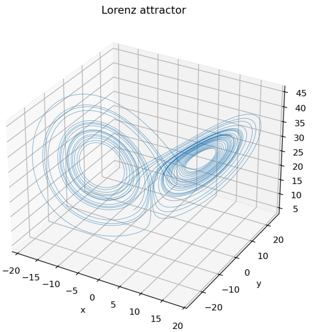
# Prelecture discussion

Before Lecture 6, run `logisticMapSensitivity.py` and explore the logistic map at $r = 4$.

- Start with two initial conditions that differ by only $10^{-8}$.
- Plot both time series and also the distance $|x_n - y_n|$ versus $n$.
- Change $r$ between values such as 3.2, 3.5, 3.8, and 4.0.

Questions to reflect on:

- For which values of $r$ do nearby initial conditions stay close, and for which values do they quickly separate?
- If the rule is deterministic, why can long-time prediction still fail?



# Lecture goals

In this lecture we will:

- explain what is usually meant by chaos in deterministic dynamical systems;
- define and interpret sensitive dependence on initial conditions;
- use the logistic map as the main discrete example, especially the case $r = 4$;
- connect local instability with the global mechanism of stretching and folding;
- revisit the standard map as a higher-dimensional example with ordered and irregular regions;
- motivate continuous-time chaos through Poincaré maps and the Lorenz system;
- discuss what numerical experiments can and cannot tell us about chaotic dynamics.

# From bifurcations to chaos

# Where we are in the course

The previous lectures gave us the language needed for chaos:

1. **Lecture 2:** discrete dynamical systems, iterated maps, fixed points, linearization.
2. **Lecture 3:** the logistic family $x_{n+1} = rx_n(1 - x_n)$ and the bifurcation diagram.
3. **Lecture 4:** bifurcations and qualitative change, especially period doubling.
4. **Lecture 5:** higher-dimensional maps and the standard map.

Natural next question:

**Core question**

After period doubling, irregular phase portraits, and increasingly complicated long-time behavior, what should we call the new regime and how should we analyze it?

# What should "chaos" mean?

A chaotic system is *not* one with no rule. It is usually the opposite:

- the evolution law is completely deterministic;
- the state stays in a bounded region of phase space;
- nevertheless, long-term prediction becomes practically impossible.

Typical signatures are:

- sensitive dependence on initial conditions,
- aperiodic long-time behavior,
- strong mixing or repeated stretching and folding,
- coexistence of local instability and global boundedness.

There is no single universal definition used in every context, but these ideas recur across the subject.

# A useful working definition for this course

For this course, a practical working description is:

## Working description of chaos

A deterministic dynamical system displays chaos when nearby initial conditions separate rapidly, the orbit does not settle to a stable equilibrium or periodic orbit, and the dynamics remains confined to a bounded region with sustained irregular behavior.

Why this definition is useful for us:

- it is close to what we can observe numerically;
- it fits our main examples: logistic map, standard map, Lorenz equations;
- it emphasizes predictability rather than only geometric complexity.

In more advanced courses one also studies formal notions such as Devaney chaos, Li–Yorke chaos, topological entropy, and SRB/invariant measures.

# Sensitive dependence on initial conditions

Let $f : X \to X$ be a map on a metric space $(X, d)$.

## Sensitive dependence

We say that $f$ has sensitive dependence if there exists a scale $\delta > 0$ such that for every point $x \in X$ and every neighborhood of $x$, we can find $y$ arbitrarily close to $x$ and an iterate $n \geq 0$ with

$$d(f^n(x), f^n(y)) > \delta.$$

Interpretation:

- initial errors can be tiny;
- after enough iterations, the resulting states become macroscopically different;
- deterministic evolution does *not* imply practical predictability.

# Predictability horizon and Lyapunov growth

Let $x_n$ and $y_n$ be two trajectories of the same system, and define their separation at step $n$ by

$$\delta x_n = y_n - x_n.$$

If the motion is locally unstable, then for some time one often observes the approximate growth law

$$\|\delta x_n\| \approx e^{\lambda n} \|\delta x_0\|,$$

with growth rate $\lambda > 0$.
Then a rough prediction horizon is obtained from

$$\|\delta x_n\| \approx \varepsilon_{\mathrm{tol}} \quad \Longrightarrow \quad n_{\mathrm{pred}} \approx \frac{1}{\lambda} \log\left(\frac{\varepsilon_{\mathrm{tol}}}{\|\delta x_0\|}\right).$$

Consequences:

- a more accurate measurement extends prediction only *logarithmically*;
- positive Lyapunov exponent means information is lost steadily over time;
- chaos is therefore strongly connected with the limits of measurement and computation.

# Chaos in one-dimensional maps

# Local instability is not yet chaos

An unstable fixed point by itself is not enough.
Example: for the linear map $u_{n+1} = au_n$ with $|a| > 1$,

- nearby points separate exponentially;
- but the orbit simply escapes to infinity unless some global mechanism keeps it bounded.

So true chaotic behavior requires two ingredients:

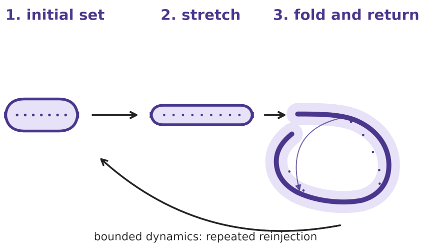
1. **stretching:** nearby states separate very fast;
2. **folding/reinjection:** trajectories are returned to a bounded region.

This is why nonlinear maps can be chaotic while simple linear growth is not.

# Stretching and folding: the geometric mechanism

- Stretching creates sensitivity by amplifying small differences.
- Folding keeps the orbit from escaping and repeatedly reuses the same bounded region.
- Repetition of these two steps produces complicated invariant sets and mixing.

This picture is the common geometric mechanism behind many chaotic systems.



# Route to chaos in the logistic family

Recall the family

$$x_{n+1} = f_r(x_n) = rx_n(1 - x_n), \qquad 0 \leq x_n \leq 1, \quad 0 \leq r \leq 4.$$

What we already know:

- for small $r$, orbits approach a stable fixed point;
- at $r = 3$, the nonzero fixed point loses stability through a period-doubling bifurcation;
- then a cascade of doublings produces periods $2, 4, 8, 16, \ldots$.

The onset of chaos occurs near

$$r_\infty \approx 3.56995.$$

Beyond that value, chaotic regimes coexist with periodic windows; for example, near $r \approx 1 + \sqrt{8}$ a stable period-three window appears.

# Why the case $r = 4$ is special

At the extreme parameter value $r = 4$ we obtain

$$x_{n+1} = 4x_n(1 - x_n), \qquad x_n \in [0, 1].$$

Reasons this case is especially useful:

- the interval $[0, 1]$ is invariant;
- the dynamics is already fully nonlinear and strongly expanding in large regions;
- the map is related to the doubling map, which makes the mechanism of chaos almost explicit;
- it gives a clean example of deterministic unpredictability in one dimension.

So although the logistic family becomes chaotic already before $r = 4$, the value $r = 4$ is the simplest place to see the mechanism analytically.

# The logistic map at $r = 4$ and the doubling map

Set

$$x_n = \sin^2(\pi\theta_n), \qquad \theta_n \in [0, 1].$$

Then

$$x_{n+1} = 4x_n(1 - x_n) = 4\sin^2(\pi\theta_n)\cos^2(\pi\theta_n) = \sin^2(2\pi\theta_n).$$

Hence

$$x_{n+1} = \sin^2(2\pi\theta_n).$$

So the logistic map at $r = 4$ is the image of the map

$$\theta_{n+1} = 2\theta_n \pmod 1$$

under the projection $x = \sin^2(\pi\theta)$.

## Important point

This is a semiconjugacy: the doubling map on the circle provides a transparent model for the chaotic behavior of the logistic map at $r = 4$.

# Binary expansion: why nearby data are quickly forgotten

Write the initial angle in binary form:

$$\theta_0 = 0.b_1 b_2 b_3 b_4 \dots$$

with digits $b_j \in \{0, 1\}$.
For the doubling map,

$$\theta_{n+1} = 2\theta_n \pmod 1$$

acts as a left shift on the binary expansion:

$$0.b_1 b_2 b_3 b_4 \cdots \mapsto 0.b_2 b_3 b_4 \dots$$

Interpretation:

- the first digit is discarded at each step;
- after $n$ iterates, prediction requires knowledge of the first $n$ binary digits of the initial condition;
- an error in a far-away digit becomes an $O(1)$ error after enough iterations.

This is a very concrete mathematical picture of sensitive dependence.

# Lyapunov exponent for one-dimensional maps

For a differentiable one-dimensional map $x_{n+1} = f(x_n)$, a natural growth rate is

$$\lambda(x_0) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |f'(x_j)|.$$

Interpretation:

- $\lambda < 0$: nearby orbits contract on average;
- $\lambda > 0$: nearby orbits separate exponentially on average;
- $\lambda = 0$: borderline or nonhyperbolic situation.

For the doubling map $\theta_{n+1} = 2\theta_n \pmod 1$,

$$|g'(\theta)| = 2 \qquad \Longrightarrow \qquad \lambda = \log 2.$$

Because the logistic map at $r = 4$ is semiconjugate to the doubling map, its typical Lyapunov exponent is also positive, in fact equal to $\log 2$.

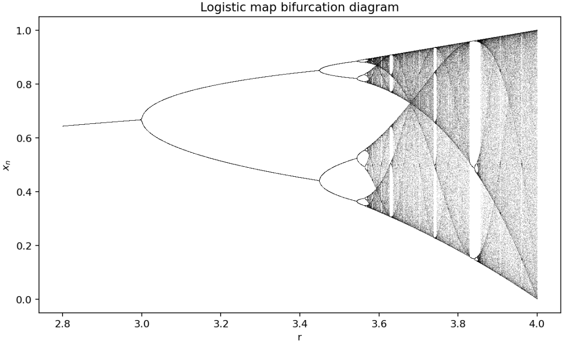
# Unpredictable trajectories, stable statistics

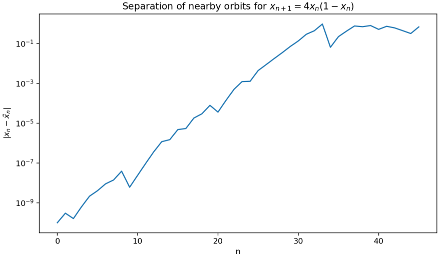
Chaos does not mean that *every* observable becomes hopeless.
For the logistic map at $r = 4$:

- a single trajectory is highly unpredictable in detail;
- but long runs still show reproducible global patterns;
- for example, histograms built from long trajectories look similar for many initial conditions.

This is an important modelling lesson in physics: even when trajectory-by-trajectory prediction fails, one can still study robust averages, histograms, or other statistical observables.

# Numerical evidence in the logistic family




Left: bifurcation diagram showing period doubling, onset of chaos, and periodic windows.

Right: separation of two nearby initial conditions for the logistic map at $r = 4$.

# Higher-dimensional discrete chaos

# From local linearization to global geometry

In higher dimension, local linearization still matters:

$$u_{n+1} = Df(x^*)u_n + \text{higher-order terms.}$$

But chaos is rarely explained by local behavior alone.

What higher dimension adds:

- saddles with stable and unstable directions;
- repeated stretching along some directions and contraction along others;
- intersections of invariant manifolds and complicated phase-space geometry;
- coexistence of ordered and irregular regions in the same system.

So in higher dimension, global geometry becomes as important as eigenvalues.

# Revisiting the Chirikov standard map

The standard map from the previous lecture is

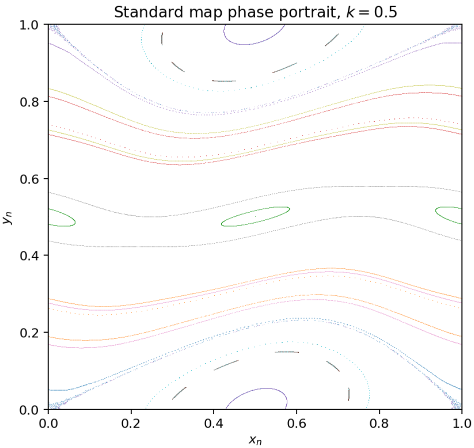
$$x_{n+1} = x_n + y_n + \frac{k}{2\pi}\sin(2\pi x_n) \pmod 1, \qquad y_{n+1} = y_n + \frac{k}{2\pi}\sin(2\pi x_n) \pmod 1.$$

Key facts:

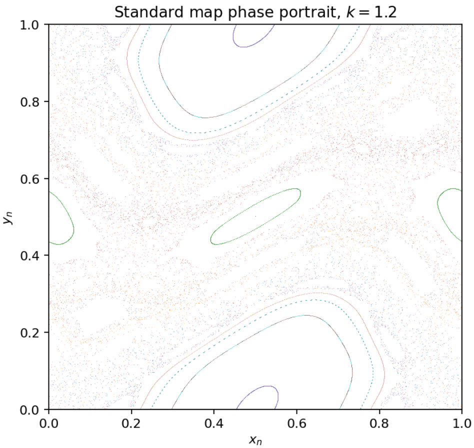
- the state is $(x_n, y_n) \in \mathbb{T}^2$;
- the parameter $k$ controls the strength of the nonlinearity;
- the Jacobian has determinant 1, so the map is area preserving;
- therefore it does not have asymptotically attracting fixed points like the logistic map does.

Nevertheless, for suitable $k$, the phase portrait contains regions that look strongly irregular.

# Ordered islands and chaotic sea in the standard map




$k = 0.5$: a large part of phase space still looks ordered.




$k = 1.2$: irregular regions occupy a much larger part of phase space.

# Dissipative versus conservative chaos

It is useful to distinguish two broad classes.

## Dissipative chaos

Volumes contract on average. Orbits may converge to a strange attractor. Examples: logistic map, Hénon map, Lorenz attractor.

## Conservative chaos

Phase-space volume is preserved. There may be chaotic regions, but not asymptotically attracting sets of the same kind. Examples: standard map, Hamiltonian systems, kicked rotor models.

So “chaos” is not a single geometry. The same phenomenon of sensitivity can arise in both dissipative and conservative settings.

# Why continuous time matters

# Why we still need continuous-time systems

So far the cleanest examples were discrete. But in physics and engineering, many models are naturally continuous:

$$\dot{x}(t) = F(x(t)).$$

Examples:

- mechanical systems,
- fluid and atmospheric models,
- chemical kinetics,
- electrical circuits,
- control systems.

So we need to understand how the language of chaos extends from iterated maps to flows.

# From a continuous flow to a discrete map

Discrete maps often arise naturally from continuous-time systems.
Two common constructions are:

1. **Stroboscopic map:** sample the state every forcing period.
2. **Poincaré return map:** intersect trajectories with a transversal section and record successive returns.

Why this matters:

- a periodic orbit of the flow becomes a fixed point of the return map;
- stability of the flow can be studied via the derivative of the return map;
- continuous-time chaos is often analyzed by reducing the problem to a discrete map.

This is exactly the conceptual bridge between our discrete examples and the Lorenz system.

# The Lorenz equations

A classical example of continuous-time chaos is the Lorenz system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = x(\rho - z) - y, \qquad \dot{z} = xy - \beta z.$$

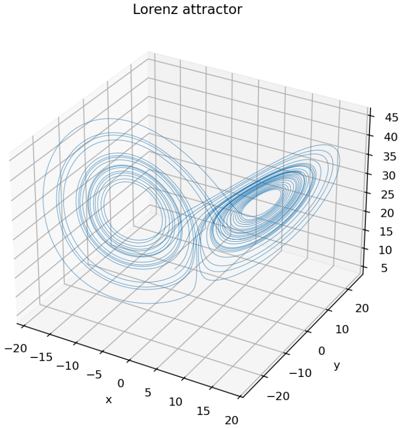
Classical parameter choice:

$$(\sigma, \rho, \beta) = (10, 28, 8/3).$$

Interpretation:

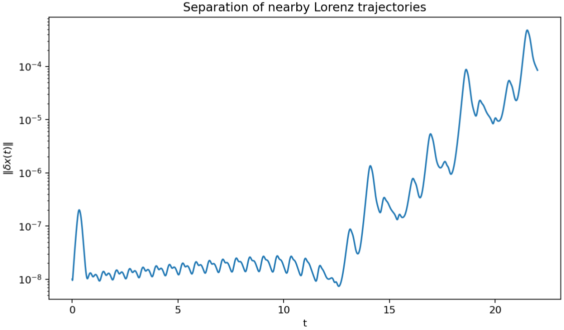
- $x$ is related to convective motion,
- $y$ measures temperature differences,
- $z$ measures distortion of the vertical temperature profile.

Although the model is very low dimensional, it already produces sensitive dependence and a complicated invariant set.




The trajectory never settles to a stable equilibrium or a simple periodic orbit, but it remains confined to a bounded butterfly-shaped region of phase space.

# Sensitive dependence in the Lorenz system




Two trajectories with initial conditions differing by only $10^{-8}$ remain close for a while and then separate rapidly.

This is the continuous-time analogue of the exponential separation we saw for the logistic map.

After the separation saturates at the attractor scale, detailed long-time prediction is lost.

# Numerical exploration and modelling lessons

# Numerical issues: what to trust and what not to trust

Chaotic dynamics creates special numerical difficulties.

- **Finite precision:** in a computer, trajectories of discrete maps eventually become periodic because only finitely many states are representable.
- **Long-term trajectory prediction:** pointwise prediction is unstable when the Lyapunov exponent is positive.
- **Transient length:** one must discard transients before deciding what the asymptotic regime looks like.
- **Time step / integrator dependence:** for ODEs, poor integration can create fake chaos or destroy real structure.
- **Robust quantities:** in chaotic systems, invariant sets, return maps, and statistical observables are often more reliable than a single orbit over a very long time.

# Suggested computational experiments

1. Build a bifurcation diagram for the logistic map and zoom near $r_\infty \approx 3.56995$.
2. For a fixed parameter such as $r = 4$, estimate

$$\lambda_n(x_0) = \frac{1}{n} \sum_{j=0}^{n-1} \log |f'(x_j)|$$

   for several initial conditions.
3. Compare two nearby logistic-map initial conditions and record the first time their distance exceeds a given threshold.
4. Re-run the standard map for several values of $k$ and compare ordered islands with irregular regions.
5. Integrate the Lorenz equations from two nearby initial conditions and estimate a finite-time growth rate.
6. For stronger students: construct a simple Poincaré section for the Lorenz system and compare it with a one-dimensional map.

# Take-away message

- Chaos is deterministic but practically unpredictable long-time behavior.
- Sensitive dependence means that tiny initial errors can become macroscopic after enough time.
- In discrete maps, the basic mechanism is stretching plus folding.
- The logistic map at $r = 4$ is a clean model because it is closely related to the doubling map and has positive Lyapunov exponent.
- Higher-dimensional systems such as the standard map show that ordered and chaotic regions can coexist.
- Continuous-time chaos is often studied through return maps; the Lorenz system is the classical example.
- In computation, pointwise trajectories are fragile, but statistical and geometric features can still be robust.

# Suggested bibliography

For students who want to read more:


- R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Westview Press, 2003.
- K. T. Alligood, T. D. Sauer, J. A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer, 1996.
- S. H. Strogatz, *Nonlinear Dynamics and Chaos*, 2nd ed., CRC Press, 2015.
- E. N. Lorenz, “Deterministic Nonperiodic Flow,” *Journal of the Atmospheric Sciences*, 20(2), 130–141, 1963.
- A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, 2nd ed., Springer, 1992.


A useful reading strategy is to compare one discrete example (logistic or tent map) with one continuous-time example (Lorenz or Rössler) and focus on the common mechanism rather than the notation.

# Acknowledgment

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All mathematical, pedagogical, and course-specific content remains under the responsibility of the lecturer.