Lecture 5: Higher-Dimensional Discrete Dynamical Systems

Modelling Complex Systems (Spring 2026)

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Prelecture discussion

Before Lecture 5, run `ChirikovStandardMap.py` and vary the parameter k.

Questions:

- Which regions look ordered?
- Which orbits appear to stay on curves?
- Which regions begin to look mixed or irregular as k increases?

Lecture goals

- Define higher-dimensional discrete dynamical systems.
- Explain fixed points and linearization in dimension $d > 1$.
- Use eigenvalues of the Jacobian to classify local stability.
- Extend the notion of periodic orbit to higher dimension.
- Study the Chirikov standard map as a basic nonlinear map on the torus.

Why go beyond one dimension?

Many systems require several state variables.

Examples:

- position and momentum,
- prey and predator,
- population in several age classes,
- several interacting agents.

In one dimension, the local picture is controlled by one number $f'(x^*)$.
In higher dimension, it is controlled by a whole matrix.

Higher-dimensional discrete dynamical systems

We now study maps of the form

$$x_{n+1} = f(x_n), \qquad x_n \in \mathbb{R}^d.$$

Here:

- x_n is the state at time n,
- f is the update rule,
- the orbit is obtained by repeated iteration of the same map.

Examples of state vectors

If $x_n = (x_n^{(1)}, x_n^{(2)})$, the coordinates may represent different physical quantities.

Typical examples in applications are:

- angle and angular momentum,
- two coupled populations,
- two spatial patches with migration,
- two interacting variables in a control system.

Fixed points in higher dimension

The definition is exactly the same as in one dimension.

Definition

A point $x^* \in \mathbb{R}^d$ is a **fixed point** if
$$f(x^*) = x^*.$$

If the orbit starts at x^*, then it remains there for all time.

Shifting the fixed point to the origin

To study the dynamics near a fixed point x^*, write

$$u_n = x_n - x^*.$$

Then the question becomes:

- if u_0 is small, does u_n go to 0?
- does it grow?
- does it oscillate or rotate while evolving?

Linearization near a fixed point

If f is differentiable, then near x^* we have the approximation

$$u_{n+1} = Df(x^*)u_n + \text{higher-order terms.}$$

Here $Df(x^*)$ is the **Jacobian matrix** of the map at the fixed point.

This is the higher-dimensional analogue of replacing a one-dimensional map by its derivative.

Connection with Lecture 2

In Lecture 2 we already studied linear maps

$$u_{n+1} = Au_n.$$

Therefore the local behavior of a nonlinear map near a fixed point is governed by the matrix

$$A = Df(x^*).$$

So the key local questions are now questions about eigenvalues and eigenvectors.

Why eigenvalues matter

For the linear map

$$u_{n+1} = Au_n,$$

the eigenvalues of A determine local growth, decay, and oscillation.

Main principles:

- if all eigenvalues satisfy $|\lambda| < 1$, the origin is attracting,
- if some eigenvalue satisfies $|\lambda| > 1$, there is an unstable direction,
- negative or complex eigenvalues may produce oscillatory behavior.

Geometric meaning of eigenvectors

If the initial condition lies in an eigenvector direction v, then

$$Av = \lambda v.$$

So one iteration only rescales that direction by λ.

Interpretation:

- eigenvectors identify preferred directions,
- $|\lambda|$ measures contraction or expansion,
- the sign or argument of λ tells us whether the motion changes side or rotates.

Real eigenvalues: typical local pictures

Suppose the eigenvalues are real.

Then the local picture often falls into one of these cases:

- both eigenvalues inside the unit circle: attraction,
- both eigenvalues outside the unit circle: repulsion,
- one inside and one outside: saddle,
- negative eigenvalues: alternating behavior along the corresponding directions.

Complex eigenvalues

If the matrix has a complex conjugate pair

$$\lambda = re^{\pm i\theta},$$

then the dynamics combines rotation and scaling.

In the linear picture:

- $r < 1$ gives a contracting spiral-type behavior,
- $r > 1$ gives an expanding spiral-type behavior,
- $r = 1$ is a borderline case in which linearization alone is not enough to conclude attraction or repulsion.

A simple linear example

Consider

$$u_{n+1} = \begin{pmatrix} 0.6 & 0 \\ 0 & -0.4 \end{pmatrix} u_n.$$

The eigenvalues are 0.6 and -0.4.

Therefore:

- both directions are contracting,
- one direction approaches monotonically,
- the other approaches with sign alternation,
- the origin is attracting.

A saddle example

Consider

$$u_{n+1} = \begin{pmatrix} 1.2 & 0 \\ 0 & 0.5 \end{pmatrix} u_n.$$

The eigenvalues are 1.2 and 0.5.

Hence one direction is unstable and one is stable.

This is the simplest example of a **saddle**, which is a central object in higher-dimensional dynamics.

Periodic orbits in higher dimension

The definition of periodic orbit is unchanged.

Definition

A point x^* has period p if

$$f^p(x^*) = x^*, \qquad f^j(x^*) \neq x^* \quad \text{for } 1 \leq j < p.$$

So a periodic orbit is a fixed point of the iterate f^p.

Stability of a periodic orbit

To study a periodic orbit, we linearize the return map.

If the orbit is

$$x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto x_0,$$

then the derivative of the return map is

$$D(f^p)(x_0) = Df(x_{p-1}) \cdots Df(x_1) Df(x_0).$$

The eigenvalues of this matrix determine the local stability of the periodic orbit.

Why higher dimension is richer

In one dimension, cobweb diagrams give a simple geometric picture.

In two dimensions and higher, we may encounter:

- saddles with stable and unstable directions,
- families of periodic orbits,
- invariant curves,
- coexistence of ordered and irregular regions.

The geometry of phase space becomes much more important.

Chirikov standard map

Motivation: the kicked rotor

The Chirikov standard map can be understood as a discrete-time model of a **kicked rotor** (that is, an angle variable).

Idea:

- a rotor evolves freely between kicks,
- at fixed times it receives a periodic impulse,
- if we observe the system immediately after each kick, the continuous evolution is reduced to a map.

This makes the model a natural example for this course because it connects:

- a physical modeling idea,
- a nonlinear map in dimension two,
- mathematical analysis,
- and computational exploration of phase portraits.

The Chirikov standard map

We now study a two-dimensional nonlinear map on the torus:

$$y_{n+1} = y_n + \frac{k}{2\pi} \sin(2\pi x_n) \pmod 1,$$

$$x_{n+1} = x_n + y_{n+1} \pmod 1.$$

The state is (x_n, y_n) and the parameter k controls the strength of the nonlinearity.

Why this example is useful

The standard map is a basic model because it is:

- simple to define,
- parameter dependent,
- genuinely two-dimensional,
- rich enough to display periodic, quasiperiodic, and more irregular-looking behavior.

It is therefore a very good example for building geometric intuition.

Fixed points of the standard map

A fixed point satisfies

$$x_{n+1} = x_n, \qquad y_{n+1} = y_n \qquad \text{mod } 1.$$

From the second equation we need

$$y^* = 0.$$

Then the first equation requires

$$\sin(2\pi x^*) = 0,$$

so on the torus we get the fixed points

$$(x^*, y^*) = (0, 0), \qquad \left(\frac{1}{2}, 0\right).$$

Jacobian matrix of the standard map

Ignoring the modulo operation when taking derivatives, we obtain

$$Df(x,y) = \begin{pmatrix} 1 + k\cos(2\pi x) & 1 \\ k\cos(2\pi x) & 1 \end{pmatrix}.$$

Its determinant is

$$\det Df(x,y) = 1.$$

So the map is area preserving.

A consequence of area preservation

Since

$$\det Df(x,y) = 1,$$

the product of the eigenvalues is 1.

Therefore, at a fixed point:

- both eigenvalues cannot satisfy $|\lambda| < 1$,
- both eigenvalues cannot satisfy $|\lambda| > 1$,
- the map cannot have an asymptotically attracting fixed point.

This is already a qualitative difference from dissipative one-dimensional maps such as the logistic map.

Local stability at $(0, 0)$

At $(0, 0)$ we have $\cos(0) = 1$, so

$$Df(0,0) = \begin{pmatrix} 1+k & 1 \\ k & 1 \end{pmatrix}.$$

Its trace is

$$\operatorname{tr} Df(0,0) = 2 + k.$$

For $k > 0$, the trace is larger than 2, so this fixed point is hyperbolic and locally saddle-like.

Local stability at $\left(\frac{1}{2}, 0\right)$

At $\left(\frac{1}{2}, 0\right)$ we have $\cos(\pi) = -1$, so

$$Df\left(\frac{1}{2}, 0\right) = \begin{pmatrix} 1-k & 1 \\ -k & 1 \end{pmatrix}.$$

Its trace is

$$\operatorname{tr} Df\left(\frac{1}{2}, 0\right) = 2 - k.$$

For $0 < k < 4$, the trace lies in $(-2, 2)$, so the eigenvalues lie on the unit circle and the local behavior is rotation-like.

Periodic, quasiperiodic, and irregular-looking motion

In a phase portrait of the standard map, different kinds of motion may coexist.

- **Periodic orbit:** only finitely many points are visited.
- **Quasiperiodic orbit:** the orbit winds around and fills a curve densely.
- **Irregular-looking orbit:** the orbit appears to spread through a region in a complicated way.

The script helps us distinguish these behaviors visually.

What should we expect as k varies?

Qualitatively:

- for smaller k, a large part of the phase portrait looks ordered,
- as k increases, regular and irregular regions coexist,
- islands associated with periodic motion may remain visible,
- some trajectories appear to wander through much larger regions of phase space.

The purpose of the script is not to prove these facts rigorously, but to build geometric intuition.

Discussion

Questions for discussion

In the standard map, what information comes from local linearization near a fixed point, and what information requires a global phase portrait?

Further questions:

- Why is the standard map so different from the logistic map?
- Why can ordered and irregular-looking regions coexist in the same system?
- What is lost when we only study one orbit instead of many initial conditions at once?

Take-away message

- In higher dimension, local behavior near a fixed point is controlled by the Jacobian matrix.
- Eigenvalues determine contraction, expansion, and oscillation.
- Periodic orbits are fixed points of iterates and are studied through the derivative of the return map.
- The Chirikov standard map shows how simple two-dimensional maps can already produce very rich geometry.

Suggested computational experiments

1. Run `ChirikovStandardMap.py` for several values of k.
2. Compare a few carefully chosen initial conditions.
3. Try to identify candidate periodic islands and curve-filling orbits.
4. Relate the local behavior near the fixed points to what you see in the global phase portrait.

Suggested bibliography

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