Lecture 4: Bifurcations and Qualitative Changes in Dynamics

Modelling Complex Systems (Spring 2026)

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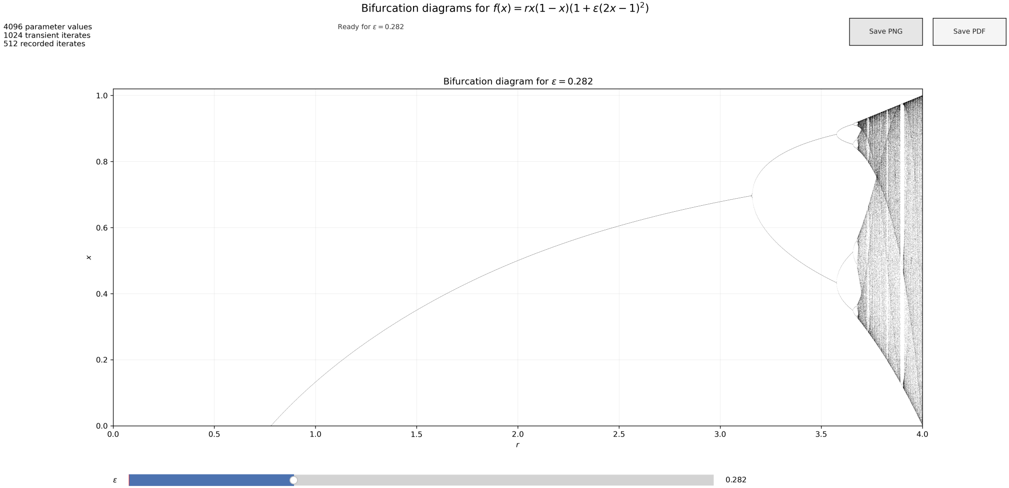
Prelecture discussion

Before Lecture 4 I suggest you to take `logisticDiagram.py` and modify it to work for other one parametric family of maps. For example, take

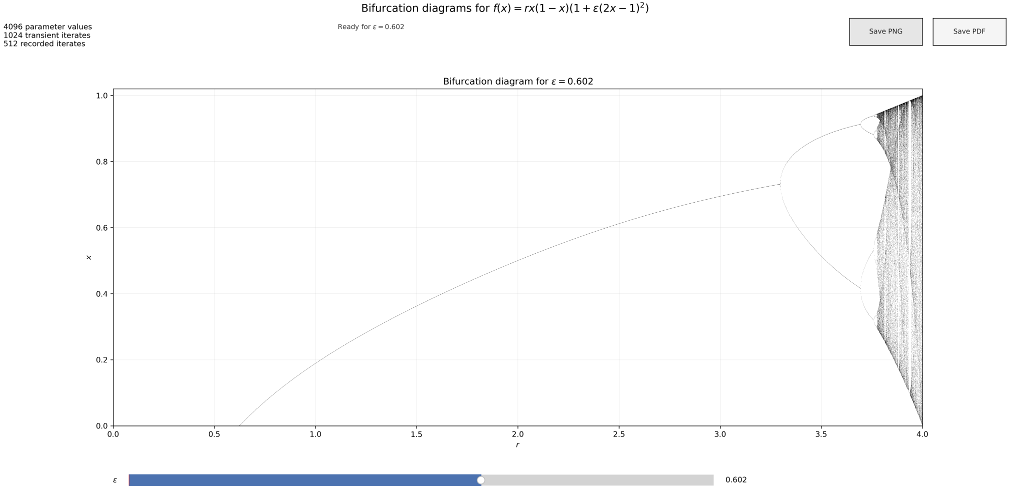
$$f(x) = rx(1-x)(1+\varepsilon(2x-1)^2),$$

for small values of ε (e.g. 0, 10^{-3}, 0.1).

Modified logistic family: $\varepsilon = 0.282$



Modified logistic family: $\varepsilon = 0.602$



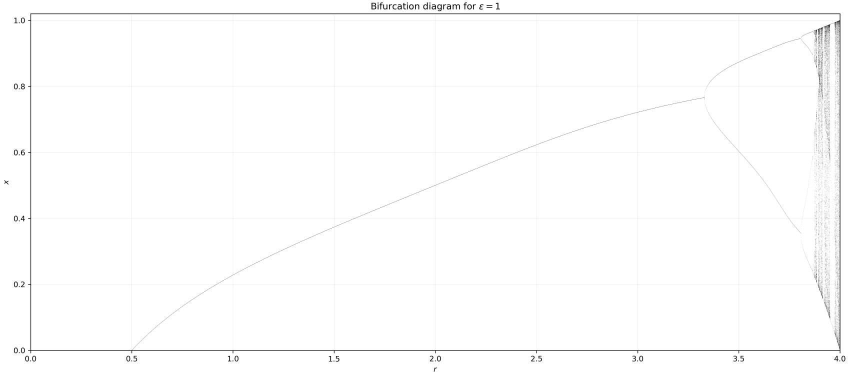
Modified logistic family: $\varepsilon = 1.000$

Bifurcation diagrams for $f(x) = rx(1-x)(1+\varepsilon(2x-1)^2)$

4096 parameter values
1024 transient iterates
512 recorded iterates

Ready for $\varepsilon = 1.000$

Save PNG
Save PDF



ε 1

Lecture goals

- Define what is meant by a parameter-dependent dynamical system.
- Explain what a bifurcation is.
- Relate bifurcations to stability changes of fixed points and periodic orbits.
- Study the main local bifurcations for one-dimensional discrete maps.
- Connect the theory with the logistic map and with numerical exploration of the standard map.

Why parameters matter

A parameter-dependent system

In applications, a dynamical system is rarely given by a single fixed rule.

We usually study a family of systems depending on a parameter μ:

$$x_{n+1} = f_\mu(x_n).$$

The parameter may represent:

- a reproduction rate,
- a coupling strength,
- an external forcing,
- a discretization or control parameter.

Qualitative versus quantitative change

Not every change in a parameter is equally important.

- A **quantitative change** modifies numerical values while preserving the same type of long-time behavior.
- A **qualitative change** modifies the structure of the dynamics itself.

Examples of qualitative change:

- a stable equilibrium becomes unstable,
- a new equilibrium appears,
- a stable periodic orbit is created,
- ordered motion gives way to more complicated behavior.

Definition of bifurcation

Informal definition

A **bifurcation** occurs when a small change in a parameter produces a qualitative change in the long-time dynamics.

What may change at a bifurcation value $\mu = \mu_c$?

- the number of fixed points,
- the stability of a fixed point,
- the existence of periodic orbits,
- the geometry of invariant sets.

What do we track as the parameter varies?

Typical objects of interest are:

- fixed points $x^*(\mu)$,
- periodic orbits,
- their stability,
- the observed long-time attractors.

In practice, we often combine:

- analytic calculations,
- local linearization,
- numerical continuation,
- direct simulation.

Local viewpoint

What should local dynamics near a fixed point look like?

Suppose x^* is a fixed point of

$$x_{n+1} = f_\mu(x_n).$$

If we start with x_0 close to x^*, we would like to understand:

- whether the orbit moves toward or away from x^*,
- whether it does so monotonically or with oscillation,
- which simple model should describe the motion nearby.

This is the basic local question behind bifurcation theory.

A theorem to keep in mind: Hartman-Grobman theorem

For a **hyperbolic** fixed point ($|f'_\mu(x)| \neq 1$), the local dynamics is as simple as the linearization.

Informally, if x^* is a fixed point and

$$|f'_\mu(x^*)| \neq 1,$$

then near x^* the nonlinear map is locally conjugate to its linear part.

So, after a suitable local change of coordinates, the dynamics looks like

$$y_{n+1} = \lambda y_n, \qquad \lambda = f'_\mu(x^*).$$

Why is this important?

This theorem explains why the derivative already tells us the local picture.

For the linear map

$$y_{n+1} = \lambda y_n,$$

we know immediately:

- if $|\lambda| < 1$, iterates contract toward 0,
- if $|\lambda| > 1$, iterates move away from 0,
- if $\lambda < 0$, the orbit alternates sides while doing so.

Hence hyperbolic fixed points have a locally robust qualitative behavior.

Question before the theorem

Questions for discussion

If a fixed point is hyperbolic for one parameter value μ_0, what should happen when we change the parameter slightly?

Should we expect:

- the fixed point to disappear immediately,
- a nearby fixed point to remain,
- its stability type to remain the same,
- or some combination of these?

Discuss this first from the fixed-point equation

$$f_\mu(x) = x.$$

Why hyperbolic fixed points persist

Write the fixed-point problem as

$$F(\mu, x) = f_\mu(x) - x = 0.$$

At a hyperbolic fixed point (μ_0, x^*) we have

$$\partial_x F(\mu_0, x^*) = f'_{\mu_0}(x^*) - 1 \neq 0.$$

Then the **Implicit Function Theorem** gives a nearby branch

$$x^*(\mu)$$

of fixed points for μ close to μ_0.

So hyperbolic fixed points do not suddenly vanish under small parameter perturbations.

Why this matters for bifurcations

If hyperbolic fixed points persist and keep their local type, then a bifurcation should not occur while hyperbolicity is preserved.

Therefore, to look for a qualitative change, we are naturally led to the *non-hyperbolic* threshold

$$|f_\mu'(x^*)| = 1.$$

This is why the condition $|f_\mu'(x^*)| = 1$ plays such a central role in one-dimensional bifurcation theory.

Beyond the Implicit Function Theorem

What if $f'_\mu(x^*) = -1$?

Questions for discussion

Consider a fixed point with

$$f'_\mu(x^*) = -1.$$

- Is this fixed point hyperbolic?
- Does the local conjugacy theorem for hyperbolic fixed points still apply?
- Can we still use the Implicit Function Theorem on

$$F(\mu, x) = f_\mu(x) - x?$$

- If the fixed point persists, what kind of qualitative change might still occur?

When the Implicit Function Theorem fails

To understand what may happen, it is useful to forget dynamics for a moment and think geometrically.

Consider an equation

$$F(x, y) = 0$$

near the point $(0, 0)$.

If

$$\partial_y F(0, 0) \neq 0,$$

then the Implicit Function Theorem says that nearby the solution set is the graph of a function

$$y = y(x).$$

But what should we expect if

$$\partial_y F(0, 0) = 0?$$

Possible local pictures for $F(x, y) = 0$

If the Implicit Function Theorem fails for solving for y, the set $F(x, y) = 0$ may look very different near $(0, 0)$.

Possible local pictures include:

- a vertical tangent, so the set is not the graph of $y = y(x)$,
- two crossing branches,
- two branches touching tangentially,
- a cusp or another singular shape,
- an isolated point.

So failure of the theorem does not tell us exactly what happens, but it tells us that the simple graph picture can break down.

A simple example

Consider

$$F(x,y) = y^2 - x.$$

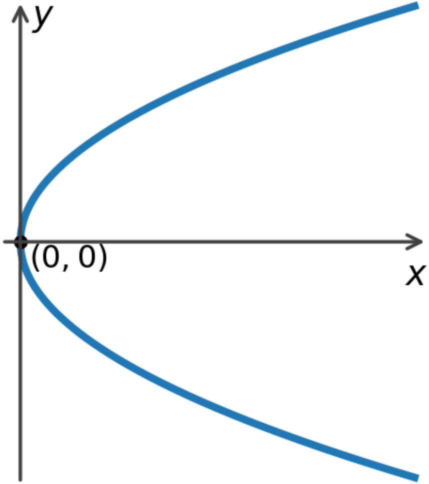
At $(0,0)$ we have

$$F(0,0) = 0, \qquad \partial_y F(0,0) = 0.$$

The solution set is

$$y = \pm\sqrt{x}.$$

Near $(0,0)$ this is not one branch $y = y(x)$, but two branches meeting at a point.



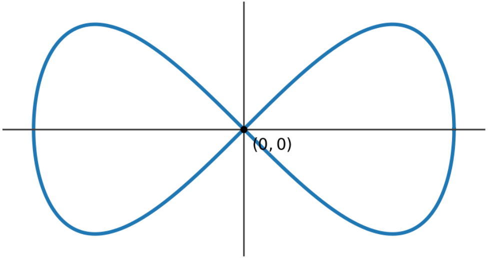
Another example: the lemniscate

Another typical picture is a self-intersection.

For example, the lemniscate of Gerono can be written as

$$F(x, y) = x^4 - x^2 + y^2 = 0.$$

At the crossing point, the solution set is not a single graph $y = y(x)$.



Bonus: How to compute at a singular point

What is a singular point?

Consider an implicit curve

$$F(x,y) = 0.$$

A point $(0,0)$ is called a **singular point** if

$$F(0,0) = 0, \qquad \partial_x F(0,0) = 0, \qquad \partial_y F(0,0) = 0.$$

At such a point, the usual Implicit Function Theorem does not help us solve locally for

$$y = y(x) \qquad \text{or} \qquad x = x(y).$$

So we need another way to detect the local branches.

A useful idea: polar coordinates

Around $(0,0)$, write

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta.$$

Then substitute into the equation:

$$F(\rho\cos\theta, \rho\sin\theta) = 0.$$

After that, remove as many powers of ρ as possible.

The remaining equation often reveals the allowed directions θ of the branches through the singular point.

Why this helps

Near the origin, the smallest power of ρ usually contains the first nontrivial geometric information.

After factoring out the largest common power of ρ, we often obtain an equation of the form

$$G(\rho, \theta) = 0,$$

with a meaningful limit as $\rho \to 0$.

Setting $\rho = 0$ in that reduced equation gives candidate angles

$$\theta = \theta_0$$

for the tangent directions of the branches.

If the reduced equation is still degenerate, we repeat the same idea again.

Applying the method to the Gerono lemniscate

Recall

$$F(x, y) = x^4 - x^2 + y^2.$$

At $(0, 0)$,

$$F(0, 0) = 0, \qquad \partial_x F(0, 0) = 0, \qquad \partial_y F(0, 0) = 0,$$

so this is a singular point.

Now write

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta.$$

Desingularizing the Gerono equation

Substituting polar coordinates gives

$$\rho^4 \cos^4 \theta - \rho^2 \cos^2 \theta + \rho^2 \sin^2 \theta = 0.$$

Factor out the largest common power of ρ:

$$\rho^2 \left(\rho^2 \cos^4 \theta - \cos^2 \theta + \sin^2 \theta\right) = 0.$$

So for $\rho \neq 0$ we reduce to

$$\rho^2 \cos^4 \theta - \cos^2 \theta + \sin^2 \theta = 0.$$

Now let $\rho \to 0$.

Angles and branches for the Gerono lemniscate

At leading order we obtain

$$-\cos^2\theta + \sin^2\theta = 0.$$

Equivalently,

$$\sin^2\theta = \cos^2\theta, \qquad \tan^2\theta = 1.$$

Hence the possible directions are

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

These are the two crossing lines

$$y = x \qquad \text{and} \qquad y = -x.$$

How many branches do we have?

In this example, the reduced equation gives four solutions for θ in $[0, 2\pi)$.

These correspond to the local directions of the curve at the singular point.

So the number of θ-solutions tells us the number of local branches passing through $(0, 0)$.

For the Gerono lemniscate:

- there are four angle solutions,
- arranged into two opposite pairs,
- therefore we see two smooth branches crossing at the origin.

Study of the period doubling

Why use f^2 for period doubling?

For the logistic map

$$f_r(x) = rx(1-x),$$

a period-2 orbit of f_r is a fixed point of the second iterate:

$$f_r^2(x) = x.$$

So near the period-doubling point $r = 3$, it is natural to study

$$F(r,x) = f_r^2(x) - x.$$

The relevant fixed point is

$$x^* = 1 - \frac{1}{r}, \qquad x^*(3) = \frac{2}{3}.$$

A singular point for $f_r^2(x) - x$

Introduce local coordinates around the bifurcation point:

$$a = r - 3, \qquad y = x - \frac{2}{3}.$$

Then we study

$$G(a, y) = f_{3+a}^2\left(\frac{2}{3} + y\right) - \left(\frac{2}{3} + y\right).$$

At the origin,

$$G(0,0) = 0, \qquad \partial_a G(0,0) = 0, \qquad \partial_y G(0,0) = 0.$$

So $(a, y) = (0, 0)$ is a singular point of the curve

$$G(a, y) = 0.$$

Why second order is enough

Since $(a, y) = (0, 0)$ is a singular point, the constant and linear terms vanish:

$$G(0,0) = 0, \qquad \partial_a G(0,0) = 0, \qquad \partial_y G(0,0) = 0.$$

Therefore the Taylor expansion starts at order 2:

$$G(a, y) = \frac{1}{2} G_{aa}(0,0)a^2 + G_{ay}(0,0)ay + \frac{1}{2} G_{yy}(0,0)y^2 + O(3).$$

So to detect the local branches, we only need the quadratic part of $f_r^2(x) - x$.

Remark on higher-order expansions

The quadratic truncation is enough to detect the tangent directions of the local branches.

If we continue the Taylor expansion to order 3, 4, or higher, then we obtain a more accurate local approximation of the curve

$$G(a, y) = 0.$$

In particular, higher-order terms improve the approximation of the actual branch geometry beyond the leading tangent information.

How to get the quadratic coefficients

Let

$$H(r,x) = f_r^2(x).$$

Then

$$G(a,y) = H\left(3+a, \frac{2}{3}+y\right) - \left(\frac{2}{3}+y\right).$$

So the quadratic Taylor coefficients of G come from derivatives of H at

$$(r,x) = \left(3, \frac{2}{3}\right).$$

The quadratic coefficients

The needed second derivatives are

$$G_{aa}(0,0) = -\frac{4}{9}, \qquad G_{ay}(0,0) = 2, \qquad G_{yy}(0,0) = 0.$$

Therefore the quadratic part is

$$\frac{1}{2}G_{aa}(0,0)a^2 + G_{ay}(0,0)ay + \frac{1}{2}G_{yy}(0,0)y^2 = -\frac{2}{9}a^2 + 2ay.$$

This is exactly the order-2 information used in the polar-coordinate analysis.

Why do the other terms vanish?

First,

$$G_y(0,0) = \partial_x H(3, 2/3) - 1 = (f_3'(2/3))^2 - 1 = (-1)^2 - 1 = 0,$$

so there is no linear term in y.

Also,

$$G_{yy}(0,0) = \partial_{xx} H(3, 2/3) = (f_3^2)''(2/3) = 0,$$

because at a fixed point of multiplier -1,

$$(f^2)''(x^*) = f''(x^*)((f'(x^*))^2 + f'(x^*)) = 0.$$

Polar coordinates near the period doubling

Write

$$a = \rho \cos\theta, \qquad y = \rho \sin\theta.$$

Expanding $G(a, y)$ near $(0, 0)$ gives

$$G(a, y) = -\frac{2}{9}a^2 + 2ay + O\Big((|a| + |y|)^3\Big).$$

Hence

$$G(\rho\cos\theta, \rho\sin\theta) = \rho^2\left(-\frac{2}{9}\cos^2\theta + 2\cos\theta\sin\theta\right) + O(\rho^3).$$

Removing the common factor ρ^2, the leading angular equation is

$$-\frac{2}{9}\cos^2\theta + 2\cos\theta\sin\theta = 0.$$

Directions of the branches

The reduced equation becomes

$$\cos\theta\left(\sin\theta - \frac{1}{9}\cos\theta\right) = 0.$$

So the possible directions are:

- $\cos\theta = 0$, that is
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2},$$
- or
$$\tan\theta = \frac{1}{9},$$
which gives the opposite pair of directions on the line
$$y = \frac{1}{9}a.$$

Thus the singular point has two different types of local branches.

Interpretation for the logistic map

The line

$$y = \frac{1}{9}a$$

is the tangent direction of the fixed-point branch

$$x^*(r) = 1 - \frac{1}{r}.$$

The vertical direction

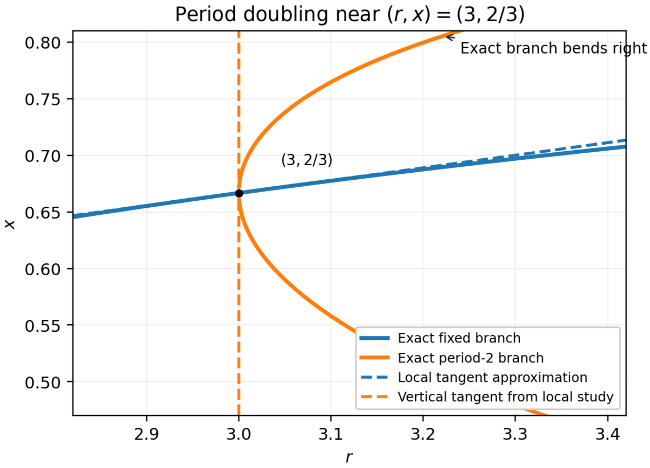
$$a = 0$$

corresponds to the new period-2 branch born at the bifurcation.

So the polar-coordinate computation detects exactly what happens at period doubling:

- one branch is the old fixed point,
- the other local directions belong to the emerging period-2 orbit.

Local picture near the period doubling



Computing the derivatives with Python

A minimal sympy script can compute the derivatives for us:

```
import sympy as sp

r, x, a, y = sp.symbols('r x a y')
f = r * x * (1 - x)
f2 = sp.expand(f.subs(x, f))
G = sp.expand(f2.subs({r: 3 + a,
                       x: sp.Rational(2, 3) + y})
              - (sp.Rational(2, 3) + y))

print(sp.diff(G, a, 2).subs({a: 0, y: 0}))
print(sp.diff(G, a, y).subs({a: 0, y: 0}))
print(sp.diff(G, y, 2).subs({a: 0, y: 0}))
```

SymPy output

Stored in:

`scriptLogisticSymbolicManipulation/logisticPeriodDoublingSympy.py`

Output:

$$G_{aa}(0,0) = -\frac{4}{9}, \qquad G_{ay}(0,0) = 2, \qquad G_{yy}(0,0) = 0.$$

Next meeting: computer lab

Next time we meet

Next time we meet, it will be in a computer lab.

There we will:

- start working on the projects,
- finalize the creation of groups,
- discuss practical questions about the project work.

Students can also bring their own computer if they want.

Summary

Take-away points

- A bifurcation is a qualitative change in long-time dynamics caused by a parameter variation.
- Hyperbolic fixed points are locally conjugate to their linearization and persist under small parameter perturbations.
- When the Implicit Function Theorem fails, singular points can produce branching, crossing, or creation of new solution curves.
- A useful way to study a singular point of $F(x, y) = 0$ is to pass to polar coordinates (ρ, θ) and factor out the lowest power of ρ.
- For the Gerono lemniscate, the number of angle solutions gives the number of local branches through the singular point.
- For the logistic map, period doubling at $r = 3$ is detected by studying $f_r^2(x) - x$ near $(r, x) = (3, 2/3)$.

Suggested reading

For students who want to read more:

- **Strogatz, *Nonlinear Dynamics and Chaos***: read the chapter on one-dimensional maps, especially fixed points, cobweb plots, bifurcation diagrams, the logistic map, and period doubling.
- **Hirsch, Smale, Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos***: read the chapters/sections on discrete dynamical systems, hyperbolic fixed points, and Hartman–Grobman / local linearization.
- **Devaney, *An Introduction to Chaotic Dynamical Systems***: read the chapters on iteration of interval maps, periodic points, and the logistic family.
- **Kuznetsov, *Elements of Applied Bifurcation Theory***: for stronger students, read the sections on local bifurcations of one-dimensional maps and period-doubling (flip) bifurcation.

Chapter numbering may vary slightly by edition, so use the chapter titles above as the main guide.