Lecture 3: The Logistic Map and the Bifurcation Diagram

Modelling Complex Systems (Spring 2026)

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Prelecture discussion

Video Logistic map

`https://www.youtube.com/watch?v=YvNnbBzJuF4`

Questions:

- What is r?
- What is represented on the left?
- What is represented on the right?

Lecture goals

- Reintroduce the logistic map as a simple nonlinear population model.
- Define fixed points for an iterated map.
- Explain local stability of a fixed point.
- Apply the stability criterion to the logistic map.
- Prepare the bifurcation diagram by tracking how long-time behavior changes with the parameter.

The logistic map

The logistic map

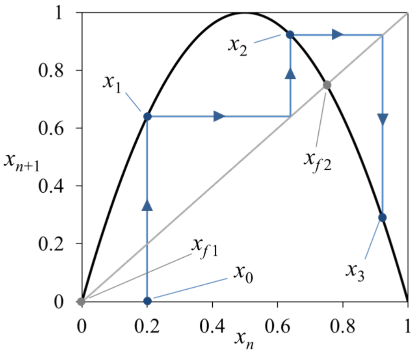
We study the discrete dynamical system

$$x_{n+1} = f_r(x_n), \qquad f_r(x) = rx(1-x),$$

where $r > 0$ is a parameter.

Interpretation:

- x_n represents the normalized population at generation n,
- reproduction is modeled by the factor r,
- competition for resources is modeled by the factor $(1 - x_n)$.



Source: Wikimedia Commons, logistic-map cobweb diagram.

Why can this model describe population dynamics?

Let N_n be the population at generation n.

A simple modeling idea is:

- if resources were unlimited, then the next generation would be proportional to the current one,
$$N_{n+1} \propto N_n;$$
- if resources are limited, then reproduction should decrease when the population gets close to a carrying capacity K;
- the simplest linear correction is the factor
$$1 - \frac{N_n}{K}.$$

Therefore one writes

$$N_{n+1} = RN_n \left(1 - \frac{N_n}{K}\right),$$

where $R > 0$ is a reproduction parameter and $K > 0$ is the carrying capacity.

Normalization and relation with the logistic map

Introduce the normalized population

$$x_n = \frac{N_n}{K}.$$

Then x_n is dimensionless, and $x_n = 1$ corresponds to the carrying capacity.

Dividing the equation

$$N_{n+1} = RN_n \left(1 - \frac{N_n}{K}\right)$$

by K, we obtain

$$x_{n+1} = Rx_n(1 - x_n).$$

So the logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

is a normalized discrete-time population model, with $r = R$.

Relation with the continuous logistic equation

A standard continuous-time model for population growth is the logistic ODE

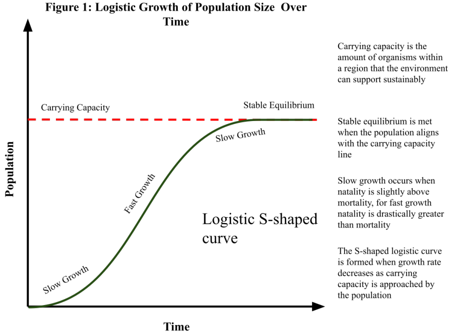
$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right),$$

where:

- $a > 0$ is the intrinsic growth rate,
- $K > 0$ is the carrying capacity.

Interpretation:

- for small N, the equation is close to exponential growth $\dot{N} \approx aN$;
- near $N = K$, growth slows down;
- at $N = K$, the growth rate is zero.



Source: Wikimedia Commons, `File:Logistic Carrying Capacity.svg`.

The logistic map as an Euler step

Apply the forward Euler method with time step $h > 0$ to

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right).$$

Then

$$N_{n+1} = N_n + haN_n\left(1 - \frac{N_n}{K}\right).$$

In terms of $x_n = N_n/K$, this becomes

$$x_{n+1} = x_n + \mu x_n(1 - x_n), \qquad \mu = ha.$$

This is not exactly $rx_n(1 - x_n)$ in the same variable, but after the rescaling $y_n = \frac{\mu}{1+\mu}x_n$, it becomes $y_{n+1} = (1 + \mu)y_n(1 - y_n)$.
So the logistic map can be seen both as a discrete population model and, up to a change of variables, as an Euler discretization of the logistic ODE.

Why this model is interesting

The rule is simple, but iteration produces rich behavior.

Questions we will ask:

- Does the population converge to an equilibrium?
- Can it oscillate between several values?
- How does the long-time behavior depend on r?
- Where does complicated or chaotic behavior appear?

State space and parameter range

For many applications we focus on

$$0 \leq x_n \leq 1.$$

Remark

If $0 \leq r \leq 4$ and $0 \leq x_n \leq 1$, then also $0 \leq x_{n+1} \leq 1$.

So for $0 \leq r \leq 4$, the interval $[0, 1]$ is invariant under the logistic map.

Fixed points

What is a fixed point?

Let $f : X \to X$ be a discrete dynamical system.

Definition

A point $x^* \in X$ is a **fixed point** if

$$f(x^*) = x^*.$$

If the orbit starts at a fixed point, then it stays there for all time:

$$x_0 = x^* \implies x_n = x^* \quad \text{for all } n \geq 0.$$

How do we compute fixed points?

Discussion question

Suppose we know the map

$$x_{n+1} = f(x_n).$$

How should we compute its fixed points?

Source: Wikimedia Commons, `File:Discussion icon.png`.

Ways to compute fixed points

Once we write the fixed-point equation

$$f(x) = x,$$

there are several possible approaches:

- **Analytic:** solve the equation explicitly.
  - Example: for the logistic map, we can solve a polynomial equation by hand.
- **Semianalytic:** use approximations or asymptotic expansions to obtain approximate fixed points.
  - Useful when an exact formula is not available but the equation has structure.
- **Numerical:** solve the equation with an algorithm such as bisection or Newton's method.
  - This is often the only practical option for complicated nonlinear maps.

Fixed points of the logistic map

To find the fixed points, solve

$$rx(1-x) = x.$$

Rearranging,

$$rx - rx^2 = x \qquad \Longleftrightarrow \qquad x\big(r(1-x) - 1\big) = 0.$$

Therefore the fixed points are

$$x_1^* = 0, \qquad x_2^* = 1 - \frac{1}{r}$$

whenever the second expression is defined.

When do these fixed points make sense?

- $x^* = 0$ is a fixed point for every r.
- $x^* = 1 - \dfrac{1}{r}$ exists for $r \neq 0$.
- If we restrict to the interval $[0, 1]$, then

$$1 - \frac{1}{r} \in [0, 1] \qquad \text{when } r \geq 1.$$

So inside the biologically relevant interval $[0, 1]$:

- for $0 < r < 1$, only $x^* = 0$ lies in $[0, 1]$,
- for $r \geq 1$, both fixed points lie in $[0, 1]$.

Stability

What does stability mean?

A fixed point is not only a solution of $f(x) = x$. We also want to know what happens to nearby initial conditions.

Informal idea

A fixed point is **stable** if points starting close to it remain close and are attracted to it under iteration.

For one-dimensional maps, the derivative gives a local test.

Local stability criterion

Let x^* be a fixed point of a differentiable map f.

Criterion

- If $|f'(x^*)| < 1$, then x^* is locally attracting.
- If $|f'(x^*)| > 1$, then x^* is locally repelling.
- If $|f'(x^*)| = 1$, the linear test is inconclusive.

Interpretation: near x^*, the map behaves approximately like multiplication by $f'(x^*)$.

Derivative of the logistic map

For
$$f_r(x) = rx(1-x),$$
we compute
$$f_r'(x) = r(1-2x).$$
We now evaluate this derivative at each fixed point.

Stability of $x^* = 0$

Since

$$f_r'(0) = r,$$

the fixed point $x^* = 0$ satisfies:

- it is locally attracting if $|r| < 1$,
- it is locally repelling if $|r| > 1$.

In the usual range $r > 0$:

- stable for $0 < r < 1$,
- unstable for $r > 1$.

Stability of $x^* = 1 - \dfrac{1}{r}$

Evaluate the derivative at the second fixed point:

$$f_r'\left(1 - \frac{1}{r}\right) = r\left(1 - 2\left(1 - \frac{1}{r}\right)\right) = 2 - r.$$

Therefore:

$$|2 - r| < 1 \qquad \Longleftrightarrow \qquad 1 < r < 3.$$

So this fixed point is locally attracting for $1 < r < 3$.

What changes at $r = 1$ and $r = 3$?

- At $r = 1$, the fixed point $x^* = 0$ loses stability and the nonzero fixed point appears in $[0, 1]$.
- For $1 < r < 3$, typical orbits are attracted to

$$x^* = 1 - \frac{1}{r}.$$

- At $r = 3$, this fixed point loses stability because

$$\left| f_r'\left(1 - \frac{1}{r}\right) \right| = 1.$$

This is the beginning of the bifurcation picture.

Periodic orbits

What is a periodic orbit?

Let $f : X \to X$ be a discrete dynamical system.

Definition

A point $x^* \in X$ has **period** p if
$$f^p(x^*) = x^*$$
and p is the smallest positive integer with this property.

Its orbit is
$$x^*,\ f(x^*),\ f^2(x^*),\ \ldots,\ f^{p-1}(x^*),$$
and then the same values repeat.

Relation with fixed points of iterates

Fixed points are exactly period-1 orbits.

More generally, a period-p orbit is a fixed point of the iterate f^p:

$$f^p(x) = x.$$

However, solving $f^p(x) = x$ gives:

- fixed points of f,
- points of period p,
- and possibly points of smaller period dividing p.

So after solving $f^p(x) = x$, we must exclude points with smaller period.

How do we compute periodic orbits?

To find period-p points, we solve

$$f^p(x) = x.$$

Then we check that

$$f^k(x) \neq x \qquad \text{for } 1 \leq k < p.$$

As for fixed points, there are different approaches:

- **analytic:** solve the equation explicitly when possible,
- **semianalytic:** use approximations,
- **numerical:** solve $f^p(x) - x = 0$ with root-finding methods.

Stability of a periodic orbit

Let x^* belong to a period-p orbit. We study the iterate f^p, because the orbit returns to x^* after p steps.

Criterion

The period-p orbit is locally attracting if

$$\left|(f^p)'(x^*)\right| < 1,$$

and locally repelling if

$$\left|(f^p)'(x^*)\right| > 1.$$

By the chain rule,

$$(f^p)'(x^*) = f'(x^*)f'(f(x^*))\cdots f'(f^{p-1}(x^*)).$$

So the stability of a periodic orbit is determined by the product of the derivatives along one full cycle.

Recap: numerical methods

What do we need in practice?

To compute fixed points or periodic orbits numerically, we need at least:

- a function that evaluates the map
$$x \mapsto f(x),$$
- if we want to use Newton's method efficiently, a function that evaluates the derivative
$$x \mapsto f'(x),$$
- an interval or an initial guess where we expect a solution,
- a tolerance for stopping the algorithm.

For periodic orbits of period p, we also need to compute the iterate $f^p(x)$ and, for stability, the product of derivatives along the orbit.

Pseudocode: fixed points

```
Given f, f', an initial guess x0, tolerance tol

Goal: solve g(x) = f(x) - x = 0

Newton method:
  x = x0
  repeat
    g  = f(x) - x
    gp = f'(x) - 1
    xNew = x - g/gp
    if |xNew - x| < tol: stop
    x = xNew
  return x
```

For bisection, we instead choose an interval $[a, b]$ with

$$(f(a) - a)(f(b) - b) < 0$$

and repeatedly halve the interval.

Pseudocode: periodic orbits

```
Given f, period p, initial guess x0, tolerance tol

Goal: solve gp(x) = f^p(x) - x = 0

Method:
  build the iterate fp(x) = f^p(x)
  solve fp(x) - x = 0
  check that fp(x) = x
  and fk(x) != x for k = 1, ..., p-1
```

In practice:

- use Newton or bisection on $f^p(x) - x$,
- then discard solutions with smaller period.

Pseudocode: stability test

```
Input: one point x* on a period-p orbit

multiplier = 1
x = x*
for j = 1, ..., p:
  multiplier = multiplier * f'(x)
  x = f(x)

if |multiplier| < 1:
  orbit is attracting
else if |multiplier| > 1:
  orbit is repelling
else:
  linear test is inconclusive
```

For a fixed point, this reduces to checking $|f'(x^*)|$.

Bifurcation diagram

How do we compute the bifurcation diagram?

The bifurcation diagram records the long-time values of x_n as the parameter r varies.

Basic idea:

- choose many parameter values r,
- for each r, choose an initial condition x_0,
- iterate the map forward for many steps,
- discard the transient part,
- plot the remaining iterates (r, x_n).

So the bifurcation diagram is obtained by **forward iteration in time**, not by solving one explicit formula for all long-time states.

Discussion: what are we actually computing?

Discussion question

When we iterate forward in time and discard transients, what objects are we actually detecting?

- Which orbits remain visible under forward iteration?
- Why do stable fixed points and stable periodic orbits appear?
- Why are unstable fixed points or unstable periodic orbits typically not observed?

The key point is that forward iteration selects **attracting** long-time behavior.

Source: Wikimedia Commons, `File:Discussion icon.png`.

What do we expect to see?

From the fixed-point analysis, we already expect:

- one stable branch for $0 < r < 1$ at $x = 0$,
- a new stable branch for $1 < r < 3$ at
$$x = 1 - \frac{1}{r},$$
- a qualitative change at $r = 3$.

Beyond $r = 3$, the long-time dynamics is no longer described by a stable fixed point, so we must look for periodic orbits and more complicated behavior.

Pseudocode for the bifurcation diagram

```
for each parameter value r:
  choose initial condition x

  for n = 1, ..., nTransient:
    x = f_r(x)

  for n = 1, ..., nRecorded:
    x = f_r(x)
    plot the point (r, x)
```

Important parameters in practice:

- number of r values,
- number of transient iterates to discard,
- number of iterates to record.

Code used in this lecture

In this lecture, the bifurcation diagram is generated with the Python script

`logisticDiagram.py`.

The script:

- samples many values of r in $[0, 4]$,
- iterates the logistic map forward,
- discards a transient,
- records the long-time iterates,
- and plots all points (r, x_n).

This numerical procedure lets us visualize stable long-time behavior even when explicit formulas are unavailable.