Lab 04. Topological Data Analysis and Collective Behavior

Jordi-Lluís Figueras

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1 Introduction

This lab has two themes. The first is topological data analysis: persistent homology studies how topological features appear and disappear in a family of simplicial complexes built from data. We focus on the Vietoris–Rips construction and on the interpretation of the first Betti numbers. The second is collective behavior: the Vicsek model illustrates how local alignment rules can produce coherent motion in a swarm of self-propelled particles.

Recall that, at a fixed scale parameter, the Betti number β_0 counts connected components and β_1 counts independent one-dimensional holes. In the topological data analysis exercises, all homology groups and Betti numbers are computed with coefficients in $\mathbb{Z}_2$.

2 Exercises

For every Betti-number computation in this lab, use the simplicial chain complex over $\mathbb{Z}_2$. Choose ordered bases for the chain groups C_0, C_1, and C_2, write the boundary maps

$$\partial_1 : C_1 \to C_0, \qquad \partial_2 : C_2 \to C_1,$$

as matrices over $\mathbb{Z}_2$, and compute their ranks by row reduction modulo 2. Then use

$$\beta_0 = \dim C_0 - \operatorname{rank} \partial_1, \qquad \beta_1 = \dim C_1 - \operatorname{rank} \partial_1 - \operatorname{rank} \partial_2.$$

1. Betti numbers of a Vietoris–Rips complex

Consider a point cloud at a fixed scale parameter r. Suppose that the Vietoris–Rips complex

$$K = \mathrm{VR}_r(X)$$

has the following simplices.

There are six vertices

$$a, b, c, d, e, f.$$

The one-dimensional skeleton consists of two square cycles attached along the edge cd:

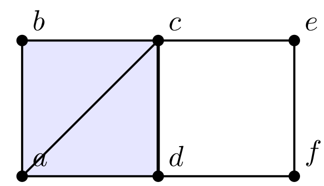
$$a - b - c - d - a, \qquad c - e - f - d - c.$$

The first square is filled by two triangular faces. More precisely, the edge ac is present and the two 2-simplices

$$[a, b, c], \qquad [a, c, d]$$

belong to the complex. The second square has no diagonal and no 2-simplex; it is only the cycle $c - e - f - d - c$.

The complex is illustrated below. The shaded triangles are 2-simplices of the complex.



Compute the Betti numbers $\beta_0(K)$ and $\beta_1(K)$.
Your solution should include:

- a list of the vertices, edges, and 2-simplices used in the computation,
- a short explanation of why β_0 has the value you obtain,
- the boundary matrices for ∂_1 and ∂_2 over $\mathbb{Z}_2$,
- the ranks of these boundary matrices and the resulting values of β_0 and β_1,
- a brief interpretation of which part of the complex contributes to a one-dimensional hole and which part does not.

2. Persistent homology of a planar point cloud

The file `points_lab04.txt` contains coordinates of 20 points in the plane. All points lie inside the unit disk. Read these points into Python and construct the Vietoris–Rips complex for a range of scale parameters r.

Use the following convention. For a fixed value of r, two points x_i and x_j form an edge if

$$\|x_i - x_j\| < r.$$

A set of $k+1$ points forms a k-simplex in the Vietoris–Rips complex if every pair of points in the set forms an edge. Equivalently, the Vietoris–Rips complex is the clique complex of the graph obtained from the distance threshold r.

For computing β_0 and β_1, it is enough to construct:

- the vertices, which are the 20 points,
- the edges, given by pairs at distance less than r,
- the triangular 2-simplices, given by triples whose three pairwise distances are all less than r.

For each value of r, compute $\beta_0(r)$ and $\beta_1(r)$ using the ranks of the boundary maps over $\mathbb{Z}_2$, as described above. Produce two plots:

$$r \mapsto \beta_0(r), \qquad r \mapsto \beta_1(r).$$

Choose a range of values of r that shows the important changes in the topology. Since the complex changes only when r crosses a distance between two points, you may either sample r on a fine grid or use the sorted list of pairwise distances to choose representative values.

Your solution should include:

- a plot of the point cloud,
- a precise explanation of how you construct the Vietoris–Rips complex from the point cloud for a given value of r,

- a description of how the boundary matrices ∂_1 and ∂_2 are assembled over $\mathbb{Z}_2$,
- the plots of $\beta_0(r)$ and $\beta_1(r)$,
- a short interpretation of the main features that appear and disappear as r increases.

3. Order in the Vicsek model

The Vicsek model is a simple model for collective motion. It describes self-propelled particles that move with constant speed and repeatedly align their direction with nearby particles.

Consider N particles in the periodic square $[0,1)^2$. Particle i has position $x_i(t) \in [0,1)^2$ and direction angle $\theta_i(t) \in [0, 2\pi)$ at time t. Let

$$u_i(t) = (\cos\theta_i(t), \sin\theta_i(t))$$

be the corresponding unit velocity direction. At each time step, particle i first computes the average direction of all particles within interaction radius R, using the periodic distance d_{per} on $[0,1)^2$. Then it adds noise and moves forward:

$$\theta_i(t+1) = \arg\left(\sum_{j:\, d_{\text{per}}(x_i(t), x_j(t)) < R} e^{\mathrm{i}\theta_j(t)}\right) + \eta\xi_i(t),$$

$$x_i(t+1) = x_i(t) + v_0(\cos\theta_i(t+1), \sin\theta_i(t+1)) \quad \text{mod } 1.$$

Here $\xi_i(t)$ are independent random variables uniformly distributed in $[-1/2, 1/2]$, and η controls the noise strength. Use noise values in the range

$$0 \le \eta \le 2\pi.$$

Implement the Vicsek model with periodic boundary conditions. Use, for example,

$$N = 500, \qquad v_0 = 0.03, \qquad R = 0.1,$$

and compare several values of the noise, such as

$$\eta = 0, \qquad 0.05, \qquad 0.1, \qquad \frac{\pi}{2}, \qquad \pi, \qquad 2\pi.$$

Start from random initial positions and random initial directions. Run the model long enough to observe the transient and the later behavior. For the snapshots, use a moderate time such as $t = 1000$.

The main global quantity is the polarization, or order parameter,

$$\Phi(t) = \left|\frac{1}{N}\sum_{i=1}^{N} u_i(t)\right|.$$

It satisfies $0 \le \Phi(t) \le 1$. Values close to 1 indicate strong global alignment, while values close to 0 indicate disordered motion.

Your solution should include:

- a precise description of the update rule you implemented, including the periodic boundary conditions,

- snapshots at a moderate time, for example $t = 1000$, showing both the particle positions and their velocity directions for at least two different noise values,
- plots of $\Phi(t)$ as a function of time,
- a plot of the time-averaged order parameter as a function of η,
- a discussion of how increasing noise changes the collective motion and the degree of alignment in the swarm.

After completing the exercise, answer the following conceptual question: why does the sum

$$\arg\left(\sum_j e^{\mathrm{i}\theta_j}\right)$$

represent an average direction in the Vicsek update rule?

3 What to submit

Submit:

- a single explanatory report in PDF format,
- the Python files used for the computations,
- the plots requested in the exercises,
- in case LLM tools were used, an appendix reporting the most important prompts used for the project.